

Small-time expansions for local jump-diffusion models with infinite jump activity

José E. Figueroa-López*

Yankeng Luo[†]

Cheng Ouyang[‡]

July 25, 2012

Abstract

We consider a Markov process X which is the solution of a stochastic differential equation driven by a Lévy process Z and an independent Wiener process W . Under some regularity conditions, including non-degeneracy of the diffusive and jump components of the process as well as smoothness of the Lévy density of Z outside any neighborhood of the origin, we obtain a small-time second-order polynomial expansion for the tail distribution and the transition density of the process X . Our method of proof combines a recent regularizing technique for deriving the analog small-time expansions for a Lévy process with some new tail and density estimates for jump-diffusion processes with small jumps based on the theory of Malliavin calculus, flow of diffeomorphisms for SDEs, and time-reversibility. As an application, the leading term for out-of-the-money option prices in short maturity under a local jump-diffusion model is also derived.

1 Introduction

The small-time asymptotic behavior of the transition densities of Markov processes $\{X_t(x)\}_{t \geq 0}$ with deterministic initial condition $X_0(x) = x$ has been studied for a long time, with a certain focus to consider either purely-continuous or purely-discontinuous processes. Starting from the problem of existence, there are several sets of sufficient conditions for the existence of the transition density of $X_t(x)$, denoted hereafter $p_t(\cdot; x)$. A stream in this direction is based on the machinery of Malliavin calculus, originally developed for continuous diffusions (see the monograph [25]) and, then, extended to Markov process with jumps (see the monograph [7]). This approach can also yield estimates of the transition density $p_t(\cdot; x)$ in small time t . For purely-jump Markov processes, the key assumption is that the Lévy measure of the process admits a smooth Lévy density. The pioneer of this approach was Léandre [19], who obtained the first-order small-time asymptotic behavior of the transition density for fully supported Lévy densities. This result was extended in [17] to the case where the point y cannot be reached with only one jump from x but rather with finitely many jumps, while [27] developed a method that can also be applied to Lévy measures with a nonzero singular component (see also [28] and [18] for other related results).

The main result in [19] states that, for $y \neq 0$,

$$\lim_{t \rightarrow 0} \frac{1}{t} p_t(x + y; x) = g(x; y),$$

where $g(x; y)$ is the so-called Lévy density of the process X to be defined below (see (5)). Léandre's approach consisted of first separating the small jumps (say, those with sizes smaller than an $\varepsilon > 0$) and the large jumps of the underlying Lévy process, and then conditioning on the number of large jumps by time t . Malliavin's calculus was then applied to control the resulting density given that there is no large jump. For $\varepsilon > 0$ small enough, the

*Department of Statistics, Purdue University, W. Lafayette, IN, 47906 USA (figueroa@purdue.edu). Research partially supported by the NSF Grants # DMS-0906919 and DMS-1149692.

[†]Department of Mathematics, Purdue University, W. Lafayette, IN, 47906 USA (luo7@purdue.edu).

[‡]Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, Chicago, IL 60607, USA (couyang@math.uic.edu).

term when there is only one large jump was proved to be equivalent, up to a remainder of order $o(t)$, to the term resulting from a model in which there is no small-jump component at all. Finally, the terms when there is more than one large jump were shown to be of order $O(t^2)$.

Higher order expansions of the transition density of Markov processes with jumps have been considered quite recently and only for processes with finite jump activity (see, e.g., [35]) or for Lévy processes with possibly infinite jump-activity. We focus on the literature of the latter case due to its close connection to the present work. [32] was the first work to consider higher order expansions for the transition densities of Lévy processes using Léandre's approach. Concretely, the following expansion for the transition densities $\{p_t(y)\}_{t \geq 0}$ of a Lévy process $\{Z_t\}_{t \geq 0}$ was proposed therein:

$$p_t(y) := \frac{d}{dy} \mathbb{P}(Z_t \leq y) = \sum_{n=1}^{N-1} a_n(y) \frac{t^n}{n!} + O(t^N), \quad (y \neq 0, N \in \mathbb{N}). \quad (1)$$

As in [19], the idea was to justify that each higher order term (say, the term corresponding to k large jumps) can be replaced, up to a remainder of order $O(t^N)$, by the resulting density as if there were no small-jump component. However, this approach is able to produce the correct expressions for the higher order coefficients $a_2(y), \dots$ only in the compound Poisson case (cf. [12]). The problem was subsequently resolved in [13] (see Section 6 therein as well as [12] for a preliminary related result), using a new approach, under the assumption that the Lévy density of the Lévy process $\{Z_t\}_{t \geq 0}$ is sufficiently smooth and bounded outside any neighborhood of the origin. There are two key ideas in [12, 13]. Firstly, instead of working directly with the transition densities, the following analog expansions for the tail probabilities were first obtained:

$$\mathbb{P}(Z_t \geq y) = \sum_{n=1}^{N-1} A_n(y) \frac{t^n}{n!} + t^N \mathcal{R}_t(y), \quad (y > 0, N \in \mathbb{N}), \quad (2)$$

where $\sup_{0 < t \leq t_0} |\mathcal{R}_t(y)| < \infty$, for some $t_0 > 0$. Secondly, by considering a smooth thresholding of the large jumps (so that the density of large jumps is smooth) and conditioning on the size of the first jump, it was possible to regularize the discontinuous functional $\mathbf{1}_{\{Z_t \geq x\}}$ and, subsequently, proceed to use an iterated Dynkin's formula (see Section 3.2 below for more information) to expand the resulting smooth moment functions $\mathbb{E}(f(Z_t))$ as a power series in t . (1) was then obtained by differentiation of (2), after justifying that the functions $A_n(y)$ and the remainder $\mathcal{R}_t(y)$ were differentiable in y .

The results and techniques described in the previous paragraph open the door to the study of higher order expansions for the transition densities of more general Markov models with infinite jump-activity. We take the analysis one step further and consider a jump-diffusion model with non-degenerate diffusion and jump components. Our analysis can also be applied to purely-discontinuous processes as in [19], but we prefer to consider a “mixture model” due to its relevance in financial applications where compelling empirical evidence supports models containing both continuous and jump components (see Section 6 below for detailed references in this direction). More concretely, we consider the following Stochastic Differential Equations (SDE) driven by a Wiener process $\{W_t\}_{t \geq 0}$ and an independent purely-jump Lévy process $\{Z_t\}_{t \geq 0}$ of bounded variation:

$$X_t(x) = x + \int_0^t b(X_u(x)) du + \int_0^t \sigma(X_u(x)) dW_u + \sum_{u \in (0, t]: \Delta Z_u \neq 0} \gamma(X_{u-}(x), \Delta Z_u). \quad (3)$$

Here, $\Delta Z_u := Z_u - Z_{u-} := Z_u - \lim_{s \nearrow u} Z_s$ denotes the jump Z at time u , and $b, \sigma : \mathbb{R} \rightarrow \mathbb{R}, \gamma : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are some suitable deterministic functions. The boundedness of the variation of Z is not essential for our analysis and is only imposed for the easiness of notation.

As it will be evident from our work, an important difficulty to deal with the model (3) arises from the more complex interplay of the jump and continuous components. In particular, conditioning on the first “big jump” of $\{X_s(x)\}_{s \leq t}$ leads us to consider the short-time expansions of the tail probability of a SDE with random initial value \tilde{J} , which creates important, albeit interesting, subtleties. More concretely, in the case of a Lévy process (i.e. when b, σ , and γ above are state-independent), conditioning on the first big jump naturally leads to analyzing the small-time expansion of the tail probability $\mathbb{P}(X_t^\varepsilon(x) + \tilde{J} \geq x + y)$, where $\{X_s^\varepsilon(x)\}$ stands for the “small jump” component of $\{X_s(x)\}$ (see the end of Section 2 for the terminology). This task is relatively simple to handle

since the smooth density of \tilde{J} “regularizes” the problem. In contrast, in the general local jump-diffusion model, conditioning on the first big jump leads to consider $\mathbb{P}(X_t^\varepsilon(x + \tilde{J}) \geq x + y)$, a problem that does not allow a direct application of Dynkin’s formula. Instead, to obtain the second-order expansion of the latter tail probability, we need to rely on smooth approximations of the tail probability building on the theoretical machinery of the flow of diffeomorphisms for SDEs and time-reversibility.

Under certain regularity conditions on b, σ and γ , as well as the Lévy measure ν of Z , we show the following second order expansion (as $t \rightarrow 0$) for the tail distribution of $\{X_t(x)\}_{t \geq 0}$:

$$\mathbb{P}(X_t(x) \geq x + y) = tA_1(x; y) + \frac{t^2}{2}A_2(x; y) + O(t^3), \quad \text{for } x \in \mathbb{R}, y > 0. \quad (4)$$

The assumptions required for (4) include boundedness and sufficient smoothness of the SDE’s coefficients as well as non-degeneracy conditions on $|\partial_\zeta \gamma(x, \zeta)|$ and $|1 + \partial_x \gamma(x, \zeta)|$. As in [19], the key assumption on the Lévy measure ν of Z is that this admits a density $h : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}_+$ that is bounded and sufficiently smooth outside any neighborhood of the origin. In that case, the leading term $A_1(x; y)$ depends only on the jump component of the process as follows

$$A_1(x; y) = \nu(\{\zeta : \gamma(x, \zeta) \geq y\}) = \int_{\{\zeta : \gamma(x, \zeta) \geq y\}} h(\zeta) d\zeta.$$

The second order term $A_2(x; y)$ admits a more complex (but explicit) representation, which enables us, for instance, to precisely characterize the effects of the drift b and the diffusion σ of the process in the likelihood of a “large” positive move (say, a move of size more than y) during a short time period t (see Remark 4.3 below for further details).

Once the asymptotic expansion for tail distribution is obtained, we proceed to obtain a second order expansion for the transition density function $p_t(y; x)$. As expected from taking formal differentiation of the tail expansion (4) with respect to y , the leading term of $p_t(x + y; x)$ is of the form $tg(x; y)$ for $y > 0$, where $g(x; y)$ is the so-called Lévy density of the process $\{X_t(x)\}_{t \geq 0}$ defined by

$$g(x; y) := -\frac{\partial}{\partial y} \nu(\{\zeta : \gamma(x, \zeta) \geq y\}) \quad (y > 0), \quad (5)$$

while the second order term takes the form $-\partial_y A_2(x; y) t^2/2$. One of the main subtleties here arises from attempting to control the density of $X_t(x)$ given that there is no “large” jump. To this end, we generalize the result in [19] to the case where there is a non-degenerate diffusion component. Again, Malliavin calculus is proved to be the key tool for this task.

Let us briefly make some remarks about the practical relevance of our results. Short-time asymptotics for the transition densities and distributions of Markov processes are important tools in many applications such as non-parametric estimation methods of the model under high-frequency sampling data and numerical approximations of functionals of the form $\Phi_t(x) := \mathbb{E}(\phi(X_T(x)))$. In many of these applications, a certain discretization of the continuous-time object under study is needed and, in that case, short-time asymptotics are important not only in developing such discrete-time approximations but also to determine the rate of convergence of the discrete-time proxies to their continuous-time counterparts.

As an instance of the applications referred to in the previous paragraph, a problem that has received a great deal of attention in the last few years is the study of small-time asymptotics for option prices and implied volatilities (see, e.g., [16], [10], [14], [6], [11], [31], [34], [15], [24], [13]). As a byproduct of the asymptotics for the tail distributions (4), we derive here the leading term of the small-time expansion for the arbitrage-free prices of out-of-the-money European call options. Specifically, let $\{S_t\}_{t \geq 0}$ be the stock price process and denote $X_t = \log S_t$ for each $t \geq 0$. We assume that \mathbb{P} is the option pricing measure and that under this measure the process $\{X_t\}_{t \geq 0}$ is of the form in (3). Then, we prove that

$$\lim_{t \rightarrow 0} \frac{1}{t} \mathbb{E}(S_t - K)_+ = \int_{-\infty}^{\infty} \left(S_0 e^{\gamma(x, \zeta)} - K \right)_+ h(\zeta) d\zeta, \quad (6)$$

which extends the analog result for exponential Lévy model (cf. [31] and [34]). A related paper is [21], where (6) was obtained for a wide class of multi-factor Lévy Markov models under certain technical conditions (see Theorem

2.1 therein), including the requirement that $\lim_{t \rightarrow 0} \mathbb{E}(S_t - K)_+/t$ exists in the “out-of-the-money region” and some stringent integrability conditions on the Lévy density h .

The paper is organized as follows. In Section 2, we introduced the model and the assumptions needed for our results. The probabilistic tools, such as the iterated Dynkin’s formula as well as tail estimates for semimartingales with bounded jumps, are presented in Section 3. The main results of the paper are then stated in Sections 4 and 5, where the second order expansion for the tail distributions and the transition densities are obtained, respectively. The application of the expansion for the tail distribution to option pricing in local jump-diffusion financial models is presented in Section 6. The proofs of our main results as well as some preliminaries of Malliavin calculus on Wiener-Poisson spaces are given in several appendices.

2 Setup, assumptions, and notation

Throughout, $C_b^{\geq 1}$ (resp., C_b^∞) represents the class of continuous (resp., bounded) functions with bounded and continuous partial derivatives of arbitrary order $n \geq 1$. We let $\{Z_t\}_{t \geq 0}$ be a driftless pure-jump Lévy process of bounded variation with Lévy measure ν and $\{W_t\}_{t \geq 0}$ be a Wiener process independent of Z , both of which are defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with the natural filtration $(\mathcal{F}_t)_{t \geq 0}$ generated by W and Z and augmented by all the null sets in \mathcal{F} so that it satisfies the *usual conditions* (see, e.g., Chapter I in [30]).

As stated in the introduction, in this paper we consider the following local jump-diffusion model

$$X_t(x) = x + \int_0^t b(X_u(x)) du + \int_0^t \sigma(X_u(x)) dW_u + \sum_{0 < u \leq t} \gamma(X_{u-}(x), \Delta Z_u), \quad (7)$$

where $\Delta Z_u := Z_u - Z_{u-} := Z_u - \lim_{s \nearrow u} Z_s$ denotes the jump Z at time u , and $b, \sigma : \mathbb{R} \rightarrow \mathbb{R}$ and $\gamma : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are deterministic functions satisfying suitable conditions under which (7) admits a unique solution. Typical sufficient conditions for (7) to be well-posed include linear growth and Lipschitz continuity of the coefficients b , σ , and γ (see, e.g., [3, Theorem 6.2.3], [26, Theorem 1.19]). It is convenient to write Z in terms of its Lévy-Itô representation:

$$Z_t = \sum_{0 < u \leq t} \Delta Z_u = \int_0^t \int_{\mathbb{R}_0} \zeta M(du, d\zeta).$$

Here, $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$ and $M(du, d\zeta) := \#\{u > 0 : (u, \Delta Z_u) \in du \times d\zeta\}$ is the jump measure of the process Z , which is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_0$ with mean measure $\mathbb{E}M(du, d\zeta) = du\nu(d\zeta)$. We make the following standing assumptions about Z :

(C1) The Lévy measure ν of Z has a $C^\infty(\mathbb{R} \setminus \{0\})$ strictly positive density h such that, for every $\varepsilon > 0$ and $n \geq 0$,

$$(i) \int (1 \wedge |\zeta|) h(\zeta) d\zeta < \infty, \quad \text{and} \quad (ii) \sup_{|\zeta| > \varepsilon} |h^{(n)}(\zeta)| < \infty. \quad (8)$$

Remark 2.1 Condition (8-i) is equivalent to the condition that Z is of bounded variation. This condition is not essential for our results and is imposed only for the easiness of notation. We could have taken a general Lévy process Z and replace the last term in (7) with a finite-variation pure-jump process plus a compensated Poisson integral as follows:

$$\begin{aligned} X_t(x) = & x + \int_0^t b(X_u(x)) du + \int_0^t \sigma(X_u(x)) dW_u \\ & + \int_0^t \int_{|\zeta| > 1} \gamma(X_{u-}(x), \zeta) M(du, d\zeta) + \int_0^t \int_{|\zeta| \leq 1} \gamma(X_{u-}(x), \zeta) \bar{M}(du, d\zeta), \end{aligned} \quad (9)$$

where $\bar{M}(du, d\zeta)$ is the compensated Poisson measure $M(du, d\zeta) - \nu(d\zeta)du$. Condition (8)-(ii) is actually necessary for the tail probabilities of $\{X_t(x)\}_{t \geq 0}$ to admit an expansion in integer powers of time. Indeed, even in the simplest pure Lévy case ($X_t(x) = Z_t + x$), it is possible to build examples where $\mathbb{P}(Z_t \geq y)$ converges to 0 at a fractional power of t in the absence of (8-ii) (see [22]).

Throughout the paper, the generator γ is assumed to satisfy the following conditions:

(C2-a) $\gamma(\cdot, \cdot) \in C_b^{\geq 1}(\mathbb{R} \times \mathbb{R})$ and $\gamma(x, 0) = 0$ for all $x \in \mathbb{R}$;

(C2-b) There exists a constant $\delta > 0$ such that $|\partial_\zeta \gamma(x, \zeta)| \geq \delta$, for all $x, \zeta \in \mathbb{R}$.

Both of the previous conditions were also imposed in [19]. Note that **(C2-a)** implies that, for any $\varepsilon > 0$, there exists $C_\varepsilon < \infty$ such that

$$\sup_x \left| \frac{\partial^i \gamma(x, \zeta)}{\partial x^i} \right| \leq C_\varepsilon |\zeta|, \quad (10)$$

for all $|\zeta| \leq \varepsilon$ and $i \geq 0$. Condition **(C2-b)** is imposed so that, for each $x \in \mathbb{R}$, the mapping $\zeta \rightarrow \gamma(x, \zeta)$ admits an inverse function $\gamma^{-1}(x, \zeta)$ with bounded derivatives. Note that **(C2-b)** together with the continuity of $\partial \gamma(x, \zeta) / \partial \zeta$ implies that the mapping $\zeta \rightarrow \gamma(x, \zeta)$ is either strictly increasing or decreasing for all x .

We will also require the following boundedness and non-degeneracy conditions:

(C3) The functions $b(x)$ and $v(x) := \sigma^2(x)/2$ belong to $C_b^\infty(\mathbb{R})$.

(C4) There exists a constant $\delta > 0$ such that, for all $x, \zeta \in \mathbb{R}$,

$$(i) \left| 1 + \frac{\partial \gamma(x, \zeta)}{\partial x} \right| \geq \delta, \quad (ii) \sigma(x) \geq \delta. \quad (11)$$

Remark 2.2 *Boundedness conditions of the type (C3) above are not restrictive in practice. Indeed, on one hand, extremely large values of b and σ won't typically make sense in a particular financial or physical phenomenon in mind (e.g. a large volatility value σ could hardly be justified financially). On the other hand, a stochastic model with arbitrary (but sufficiently regular) functions b and v could be closely approximated by a model with C_b^∞ functions b and v . The condition (11-i), which was also imposed in [19], guarantees the a.s. existence of a flow $\Phi_{s,t}(x) : \mathbb{R} \rightarrow \mathbb{R}, x \rightarrow X_{s,t}(x)$ of diffeomorphisms for all $0 \leq s \leq t$ (cf. [19]), where here $\{X_{s,t}(x)\}_{t \geq s}$ is defined as in (7) but with initial condition $X_{s,s}(x) = x$.*

As it is usually the case with Lévy processes, we shall decompose Z into a compound Poisson process and a process with bounded jumps. More specifically, let $\phi_\varepsilon \in C^\infty(\mathbb{R})$ be a truncation function such that $\mathbf{1}_{|\zeta| \geq \varepsilon} \leq \phi_\varepsilon(\zeta) \leq \mathbf{1}_{|\zeta| \geq \varepsilon/2}$ and let $\{Z_t(\varepsilon)\}_{t \geq 0}$ and $\{Z'_t(\varepsilon)\}_{t \geq 0}$ be independent driftless Lévy processes of bounded variation with respective Lévy densities

$$h_\varepsilon(\zeta) := \phi_\varepsilon(\zeta)h(\zeta) \quad \text{and} \quad \bar{h}_\varepsilon(\zeta) := (1 - \phi_\varepsilon(\zeta))h(\zeta). \quad (12)$$

Without loss of generality, we assume hereafter that

$$Z_t = Z_t(\varepsilon) + Z'_t(\varepsilon), \quad \text{for all } t \geq 0. \quad (13)$$

The process $Z'(\varepsilon)$, that we referred to as the small-jump component of Z , is a pure-jump Lévy process with jumps bounded by ε . In contrast, the process $Z(\varepsilon)$, hereafter referred to as the big-jump component of Z , is a compound Poisson process with intensity of jumps $\lambda_\varepsilon := \int \phi_\varepsilon(\zeta)h(\zeta)d\zeta$ and jumps $\{J_i^\varepsilon\}_{i \geq 1}$ with probability density function

$$\check{h}_\varepsilon(\zeta) := \frac{\phi_\varepsilon(\zeta)h(\zeta)}{\lambda_\varepsilon}. \quad (14)$$

Throughout the paper, $\{N_t^\varepsilon\}_{t \geq 0}$ denote the jump counting process of the compound Poisson process $\{Z_t(\varepsilon)\}_{t \geq 0}$ and $J := J^\varepsilon$ represent a random variable with density $\check{h}_\varepsilon(\zeta)$.

The following result, whose proof is presented in Appendix D, will be needed in what follows.

Lemma 2.3 *Under the conditions (C1), (C2) and (C4), the following statements hold:*

1. Let $\tilde{\gamma}(z, \zeta) := \gamma(z, \zeta) + z$. Then, for each $z \in \mathbb{R}$, the mapping $\zeta \rightarrow \tilde{\gamma}(z, \zeta)$ (equiv. $\zeta \rightarrow \gamma(z, \zeta)$) is invertible and its inverse $\tilde{\gamma}^{-1}(z, \zeta)$ (resp. $\gamma^{-1}(z, \zeta)$) is $C_b^{\geq 1}(\mathbb{R} \times \mathbb{R})$.

2. Both $\tilde{\gamma}(z, J^\varepsilon)$ and $\gamma(z, J^\varepsilon)$ admit densities in $C_b^\infty(\mathbb{R} \times \mathbb{R})$, denoted by $\tilde{\Gamma}(\zeta; z) := \tilde{\Gamma}_\varepsilon(\zeta; z)$ and $\Gamma(\zeta; z) := \Gamma_\varepsilon(\zeta; z)$, respectively. Furthermore, they have the representation:

$$\tilde{\Gamma}_\varepsilon(\zeta; z) = \check{h}_\varepsilon(\tilde{\gamma}^{-1}(z, \zeta)) \left| \frac{\partial \gamma}{\partial \zeta}(z, \tilde{\gamma}^{-1}(z, \zeta)) \right|^{-1}, \quad \Gamma_\varepsilon(\zeta; z) = \check{h}_\varepsilon(\gamma^{-1}(z, \zeta)) \left| \frac{\partial \gamma}{\partial \zeta}(z, \gamma^{-1}(z, \zeta)) \right|^{-1}. \quad (15)$$

3. The mappings $(z, \zeta) \rightarrow \mathbb{P}(\tilde{\gamma}(z, J^\varepsilon) \geq \zeta)$ and $(z, \zeta) \rightarrow \mathbb{P}(\gamma(z, J^\varepsilon) \geq \zeta)$ are $C_b^\infty(\mathbb{R} \times \mathbb{R})$.

4. The mapping $z \rightarrow u := z + \gamma(z, \zeta)$ admits a density, denoted hereafter $\bar{\gamma}(u, \zeta)$, that belongs to $C_b^{\geq 1}(\mathbb{R} \times \mathbb{R})$.

We finish this section with the definition of two important processes. Throughout the paper, we let $\{X_s(\varepsilon, \emptyset, x)\}_{s \geq 0}$ be the solution of the SDE:

$$\begin{aligned} X_s(\varepsilon, \emptyset, x) &:= x + \int_0^s b(X_u(\varepsilon, \emptyset, x)) du + \int_0^s \sigma(X_u(\varepsilon, \emptyset, x)) dW_u + \sum_{0 < u \leq s} \gamma(X_{u-}(\varepsilon, \emptyset, x), \Delta Z'_u(\varepsilon)) \\ &= x + \int_0^s b(X_u(\varepsilon, \emptyset, x)) du + \int_0^s \sigma(X_u(\varepsilon, \emptyset, x)) dW_u + \int_0^s \int \gamma(X_{u-}(\varepsilon, \emptyset, x), \zeta) M'(du, d\zeta), \end{aligned} \quad (16)$$

where hereafter M' denotes the jump measure of the small-jump component $Z'(\varepsilon)$. The law of the process (16) can be interpreted as the law of $\{X_s\}_{0 \leq s \leq t}$ conditioning on not having any “big” jumps during $[0, t]$. In other words, denoting the law of a process Y (resp. the conditional law of Y given an event B) by $\mathcal{L}(Y)$ (resp. $\mathcal{L}(Y|B)$), we have that, for each fixed $t > 0$,

$$\mathcal{L}\left(\{X_s(x)\}_{0 \leq s \leq t} \middle| N_t^\varepsilon = 0\right) = \mathcal{L}\left(\{X_s(\varepsilon, \emptyset, x)\}_{0 \leq s \leq t}\right).$$

Similarly, for a collection of times $0 < s_1 < \dots < s_n$, let $\{X_s(\varepsilon, \{s_1, \dots, s_n\}, x)\}_{s \geq 0}$ be the solution of the SDE:

$$\begin{aligned} X_s(\varepsilon, \{s_1, \dots, s_n\}, x) &:= x + \int_0^s b(X_u(\varepsilon, \{s_1, \dots, s_n\}, x)) du + \int_0^s \sigma(X_u(\varepsilon, \{s_1, \dots, s_n\}, x)) dW_u \\ &\quad + \sum_{0 < u \leq s} \gamma(X_{u-}(\varepsilon, \{s_1, \dots, s_n\}, x), \Delta Z'_u(\varepsilon)) + \sum_{i: s_i \leq s} \gamma(X_{s_i-}(\varepsilon, \{s_1, \dots, s_n\}, x), J_i^\varepsilon), \end{aligned}$$

where, as above, $\{J_i^\varepsilon\}_{i \geq 1}$ represents a random sample from the distribution $\phi_\varepsilon(\zeta)h(\zeta)d\zeta/\lambda_\varepsilon$, independent of Z' . Then, denoting the arrival jumps of $Z(\varepsilon)$ by $\tau_1 < \tau_2 < \dots$, it follows that

$$\mathcal{L}\left(\{X_s(x)\}_{0 \leq s \leq t} \middle| N_t^\varepsilon = n, \tau_1 = s_1, \dots, \tau_n = s_n\right) = \mathcal{L}\left(\{X_s(\varepsilon, \{s_1, \dots, s_n\}, x)\}_{0 \leq s \leq t}\right).$$

The previous two processes will be needed in order to implement Léandre’s approach in which the tail distribution $\mathbb{P}(X_t(x) \geq x + y)$ is expanded in powers of time by conditioning on the number of jumps of $Z(\varepsilon)$ by time t .

3 Probabilistic tools

Throughout, $C_b^n(I)$ (resp. C_b^n) denotes the class of functions having continuous and bounded derivatives of order $0 \leq k \leq n$ in an open interval $I \subset \mathbb{R}$ (resp. in \mathbb{R}). Also, $\|g\|_\infty = \sup_y |g(y)|$.

3.1 Uniform tail probability estimates

The following general result will be important in the sequel.

Proposition 3.1 *Let M be a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_0$ with mean measure $\mathbb{E}M(du, d\zeta) = \nu(d\zeta)dt$ and \bar{M} be its compensated random measure. Let $Y := Y^{(x)}$ be the solution of the SDE*

$$Y_t = x + \int_0^t \bar{b}(Y_s) ds + \int_0^t \bar{\sigma}(Y_s) dW_s + \int_0^t \int \bar{\gamma}(Y_{s-}, \zeta) \bar{M}(ds, d\zeta).$$

Assume that $\bar{b}(x)$ and $\bar{\sigma}(x)$ are uniformly bounded and $\bar{\gamma}(x, \zeta)$ is such that, for a constant $S < \infty$, $\sup_y |\bar{\gamma}(y, \zeta)| \leq S(|\zeta| \wedge 1)$, for ν -a.e. ζ . In particular, the jumps of $\{Y_t\}_{t \geq 0}$ are bounded by S . Then there exists a constant $C(S, k)$ depending on S and k , such that, for any fixed $p > 0$ and all $0 \leq t \leq 1$,

$$\mathbb{P} \left\{ \sup_{0 \leq s \leq t} |Y_s^{(x)} - x| \geq 2pS \right\} \leq C(S, k)t^p.$$

Proof. Let V_t be either $\int_0^t \bar{\sigma}(Y_s) dW_s$ or $\int_0^t \int \bar{\gamma}(Y_{s-}, z) \bar{M}(ds, dz)$. It is clear that, in either case, V_t is a martingale with its jumps bounded by S . Moreover, its quadratic variation $\langle V, V \rangle$ is given by either $\int_0^t \bar{\sigma}^2(Y_s) ds$ or $\int_0^t \int \bar{\gamma}^2(Y_s, \zeta) \nu(d\zeta) ds$ and, hence, in light of the boundedness of $\bar{\sigma}$ and the $\bar{\gamma}$, satisfies $\langle V, V \rangle_t \leq kt$, for some constant k . By equation (9) in [20], we have

$$\mathbb{P} \left\{ \sup_{0 \leq s \leq t} |V_s| \geq C \right\} \leq 2 \exp \left[-\lambda C + \frac{\lambda^2}{2} kt(1 + \exp[\lambda S]) \right], \quad \text{for all } C, \lambda > 0. \quad (17)$$

Now take $C = 2pS$ and $\lambda = |\log t|/2S$. The rest of the proof is then clear. ■

As a direct corollary of the above proposition, we have the following tail probability estimate for the small-jump component $\{X_t(\varepsilon, \emptyset, x)\}_{t \geq 0}$ of X defined in (16).

Lemma 3.2 Fix any $\eta > 0$ and a positive integer N . Then, under the Conditions **(C2-C3)** of Section 2, there exist an $\varepsilon := \varepsilon(N, \eta) > 0$ and $C := C(N, \eta) < \infty$ such that

(1) For all $t < 1$,

$$\sup_{0 < \varepsilon' < \varepsilon, x \in \mathbb{R}} \mathbb{P}(|X_t(\varepsilon', \emptyset, x) - x| \geq \eta) < Ct^N. \quad (18)$$

(2) For all $t < 1$,

$$\sup_{\varepsilon' < \varepsilon, x \in \mathbb{R}} \int_{\eta}^{\infty} \mathbb{P}\{e^{|X_t(\varepsilon', \emptyset, x) - x|} \geq s\} ds = \sup_{\varepsilon' < \varepsilon, x \in \mathbb{R}} \int_{\eta}^{\infty} \mathbb{P}\{|X_t(\varepsilon', \emptyset, x) - x| \geq \log s\} ds < Ct^N.$$

Proof. The first statement is a special case of Proposition 3.1, which can be applied in light of the boundedness conditions **(C3)** as well as the condition **(C2-a)** that ensured that the components

$$\int_0^t \sigma(X_u(\varepsilon', \emptyset, x)) dW_u, \quad \int_0^t \int \gamma(X_{u-}(\varepsilon', \emptyset, x), \zeta) \bar{M}'(du, d\zeta),$$

are martingales with their quadratic variation bounded by kt , for some constant $k > 0$. Here, \bar{M}' is the compensated Poisson measure of the small-jump component $Z'(\varepsilon')$. To prove the second statement, we keep the notation of the proof of Proposition 3.1. By (17) there exists a constant $C > 0$ such that

$$\begin{aligned} \int_{\eta}^{\infty} \mathbb{P}\{|X_t(\varepsilon, \emptyset, x) - x| \geq \log s\} ds &\leq C \int_{\eta}^{\infty} \exp \left[-\lambda \log s + \frac{\lambda^2}{2} kt(1 + \exp[\lambda \varepsilon]) \right] ds \\ &= \frac{C\eta}{(\lambda - 1)\eta^{\lambda}} \exp \left[\frac{\lambda^2}{2} kt(1 + \exp[\lambda \varepsilon]) \right]. \end{aligned}$$

Now it suffices to take $\lambda = |\log t|/2\varepsilon$ and $\varepsilon = \log \eta/2N$. ■

3.2 Iterated Dynkin's formula

We now proceed to formally state a second-order iterated Dynkin's formula for the “small-time component” of X , $\{X_t(\varepsilon, \emptyset, x)\}_{t \geq 0}$, defined in (16). In general, the n -order iterated Dynkin's formula of a Markov process Y with initial condition $Y_0 = x$ and generator L_Y takes the generic form:

$$\mathbb{E}f(Y_t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} L_Y^k f(x) + \frac{t^n}{(n-1)!} \int_0^1 (1-\alpha)^{n-1} \mathbb{E}\{L_Y^n f(Y_{\alpha t})\} d\alpha, \quad (19)$$

where as usual $L_Y^0 f = f$ and $L_Y^n f = L_Y(L_Y^{n-1} f)$, $n \geq 1$. For the general jump-diffusion process (9), (19) can be proved for $n = 1$ using Itô's formula (see [26, Theorem 1.23]), while the general case can be proved by induction under somewhat strong conditions. In what follows, we only need the second-order formula and, thus, the subsequent result will suffice for our purposes. Below, L_ε denotes the infinitesimal generator for $\{X_t(\varepsilon, \emptyset, x)\}_{t \geq 0}$ which can be written as (c.f. [26])

$$\begin{aligned} L_\varepsilon f(y) &:= \mathcal{D}_\varepsilon f(y) + \mathcal{I}_\varepsilon f(y), \quad \text{with} \\ \mathcal{D}_\varepsilon f(y) &:= \frac{\sigma^2(y)}{2} f''(y) + b(y) f'(y), \quad \mathcal{I}_\varepsilon f(y) := \int (f(y + \gamma(y, \zeta)) - f(y)) \bar{h}_\varepsilon(\zeta) d\zeta. \end{aligned} \quad (20)$$

Hereafter, $\bar{h}_\varepsilon(\zeta) := (1 - \phi_\varepsilon(\zeta))h(\zeta)$ denotes the density of h truncated outside of a ε -neighborhood of the origin. Let us remark that (20) is well-defined whenever $f \in C_b^2$ in light of condition **(C2-a)**. The proof of the following result is presented in Appendix D.

Lemma 3.3 *Under the Conditions (C1)-(C3) of Section 2, the following assertions hold:*

1. For any function f in C_b^2 ,

$$\mathbb{E}f(X_t(\varepsilon, \emptyset, x)) = f(x) + t \int_0^1 \mathbb{E}L_\varepsilon f(X_{\alpha t}(\varepsilon, \emptyset, x)) d\alpha, \quad (21)$$

where the remainder term is such that

$$\sup_{\substack{0 < t < 1, x \in \mathbb{R} \\ \varepsilon' < \varepsilon}} \left| \int_0^1 \mathbb{E}L_\varepsilon f(X_{\alpha t}(\varepsilon', \emptyset, x)) d\alpha \right| \leq K, \quad (22)$$

for a constant $K < \infty$ depending only on $\int |\zeta| \bar{h}_\varepsilon(\zeta) d\zeta$, $\|f^{(k)}\|_\infty$, $\|b^{(k)}\|_\infty$, and $\|v^{(k)}\|_\infty$ for $k = 0, 1, 2$.

2. Additionally, if $f \in C_b^4$, then

$$\mathbb{E}f(X_t(\varepsilon, \emptyset, x)) = f(x) + tL_\varepsilon f(x) + t^2 \int_0^1 (1 - \alpha) \mathbb{E}L_\varepsilon^2 f(X_{\alpha t}(\varepsilon, \emptyset, x)) d\alpha, \quad (23)$$

where the remainder term is such that

$$\sup_{\substack{0 < t < 1, x \in \mathbb{R} \\ \varepsilon' < \varepsilon}} \left| \int_0^1 (1 - \alpha) \mathbb{E}L_\varepsilon^2 f(X_{\alpha t}(\varepsilon', \emptyset, x)) d\alpha \right| \leq K, \quad (24)$$

for a constant $K < \infty$ depending only on $\int |\zeta| \bar{h}_\varepsilon(\zeta) d\zeta$, $\|f^{(k)}\|_\infty$, $\|b^{(k)}\|_\infty$, and $\|v^{(k)}\|_\infty$ for $k = 0, \dots, 4$.

4 Second order expansion for the tail distributions

We are ready to state our first main result; namely, we characterize the small-time behavior of the tail distribution of $\{X_t(x)\}_{t \geq 0}$:

$$\bar{F}_t(x, y) := \mathbb{P}(X_t(x) \geq x + y), \quad (y > 0). \quad (25)$$

As in [19], the key idea is to use the decomposition (13) by conditioning on the number of “large” jumps occurring before time t . Concretely, recalling that $\{N_t^\varepsilon\}_{t \geq 0}$ and $\lambda_\varepsilon := \int \phi_\varepsilon(\zeta) h(\zeta) d\zeta$ represent the jump counting process and the jump intensity of the large-jump component process $\{Z_t(\varepsilon)\}_{t \geq 0}$ of Z , we have

$$\mathbb{P}(X_t(x) \geq x + y) = e^{-\lambda_\varepsilon t} \sum_{n=0}^{\infty} \mathbb{P}(X_t(x) \geq x + y | N_t^\varepsilon = n) \frac{(\lambda_\varepsilon t)^n}{n!}. \quad (26)$$

The first term in (26) (when $n = 0$) can be written as

$$\mathbb{P}(X_t(x) \geq x + y | N_t^\varepsilon = 0) = \mathbb{P}(X_t(\varepsilon, \emptyset, x) \geq x + y).$$

In light of (18), this term can be made $O(t^N)$ for an arbitrarily large $N \geq 1$, by taking ε small enough. In order to deal with the other terms in (26), we use the iterated Dynkin's formula introduced in Section 3.2. The following is the main result of this section (see Appendix B for the proof). Below, h_ε and \bar{h}_ε denote the Lévy densities defined in (12), while $g(x; y)$ denotes the so-called Lévy density of the process $\{X_t(x)\}_{t \geq 0}$ defined by

$$g(x; y) := -\frac{\partial}{\partial y} \int_{\{\zeta: \gamma(x, \zeta) \geq y\}} h(\zeta) d\zeta, \quad (27)$$

for $y > 0$. In light of Lemma 2.3, g admits the representation:

$$g(x; y) = h(\gamma^{-1}(x, y)) |(\partial_\zeta \gamma)(x, \gamma^{-1}(x, y))|^{-1},$$

where $\partial_\zeta \gamma$ is the partial derivative of the function $\gamma(x, \zeta)$ with respect to its second variable.

Theorem 4.1 *Let $x \in \mathbb{R}$ and $y > 0$. Then, under the Conditions (C1-C4) of Section 2, we have*

$$\bar{F}_t(x, y) := \mathbb{P}(X_t(x) \geq x + y) = tA_1(x; y) + \frac{t^2}{2}A_2(x; y) + O(t^3), \quad (28)$$

as $t \rightarrow 0$, where $A_1(x; y)$ and $A_2(x; y)$ admit the following representations (for $\varepsilon > 0$ small enough):

$$\begin{aligned} A_1(x; y) &:= \int_y^\infty g(x; \zeta) d\zeta = \int_{\{\gamma(x, \zeta) \geq y\}} h(\zeta) d\zeta, \\ A_2(x; y) &:= \mathcal{D}(x; y) + \mathcal{J}_1(x; y) + \mathcal{J}_2(x; y), \end{aligned}$$

with

$$\begin{aligned} \mathcal{D}(x; y) &= b(x) \left(\frac{\partial}{\partial x} \int_y^\infty g(x; \zeta) d\zeta + g(x; y) \right) + b(x + y)g(x; y) \\ &\quad + \frac{\sigma^2(x)}{2} \left(\frac{\partial^2}{\partial x^2} \int_y^\infty g(x; \zeta) d\zeta + 2 \frac{\partial}{\partial x} g(x; y) - \frac{\partial}{\partial y} g(x; y) \right), \\ &\quad - \frac{\sigma(x + y)}{2} \left(\sigma(x + y) \frac{\partial}{\partial y} g(x; y) + 2\sigma'(x + y)g(x; y) \right) \\ \mathcal{J}_1(x; y) &= \int \left(\int_{y-\gamma(x, \bar{\zeta})}^\infty g(x; \zeta) d\zeta + \int_{\bar{\gamma}(x+y, \bar{\zeta})-x}^\infty g(x; \zeta) d\zeta - 2 \int_y^\infty g(x; \zeta) d\zeta \right) \bar{h}_\varepsilon(\bar{\zeta}) d\bar{\zeta}, \\ \mathcal{J}_2(x; y) &= \int \int_{y-\gamma(x, \zeta)}^\infty g(x + \gamma(x, \zeta); \zeta') d\zeta' h_\varepsilon(\zeta) d\zeta - 2 \int_y^\infty g(x; \zeta) d\zeta \int h_\varepsilon(\zeta) d\zeta. \end{aligned} \quad (29)$$

Remark 4.2 *Note that if $\text{supp}(\nu) \cap \{\zeta : \gamma(x, \zeta) \geq y\} = \emptyset$ (so that it is not possible to reach the level y from x with only one jump), then $A_1(x; y) = 0$ and $\mathbb{P}(X_t(x) \geq x + y) = O(t^2)$ as $t \rightarrow 0$. If, in addition, it is possible to reach the level y from x with two jumps, then $\mathcal{J}_2(x; y) \neq 0$, implying that $\mathbb{P}(X_t(x) \geq x + y)$ decreases at the order of t^2 . These observations are consistent with the results in [17] and [28].*

Remark 4.3 *In the case that the generator $\gamma(x, \zeta)$ does not depend on x , we get the following expansion for $\mathbb{P}(X_t(x) \geq x + y)$:*

$$\begin{aligned} &t \int_y^\infty g(\zeta) d\zeta + \frac{b(x) + b(x + y)}{2} g(y) t^2 - \left(\frac{\sigma^2(x) + \sigma^2(x + y)}{2} g'(y) + 2\sigma(x + y)\sigma'(x + y)g(y) \right) \frac{t^2}{2} \\ &\quad + \int \left(\int_{y-\gamma(\bar{\zeta})}^\infty g(\zeta) d\zeta - \int_y^\infty g(\zeta) d\zeta \right) \bar{h}_\varepsilon(\bar{\zeta}) d\bar{\zeta} t^2 \\ &\quad + \left(\int \int_{y-\gamma(\zeta)}^\infty g(\zeta') d\zeta' h_\varepsilon(\zeta) d\zeta - 2 \int_y^\infty g(\zeta) d\zeta \int h_\varepsilon(\zeta) d\zeta \right) \frac{t^2}{2} + O(t^3). \end{aligned}$$

The leading term in the above expression is determined by the jump component of the process and it has a natural interpretation: if within a very short time interval there is a “large” positive move (say, a move by more than y), this move must be due to a “large” jump. It is until the second term, when the diffusion and drift terms of the process $X(x)$ appear. If, for instance, b and σ are constants, the effect of a positive drift $b > 0$ is to increase the probability of a “large” positive move of more than y by $bg(y)t^2(1 + o(1))$. Similarly, since typically $g'(y) < 0$ when $y > 0$, the effect of a non-zero spot volatility σ is to increase the probability of a “large” positive move by $\frac{\sigma^2}{2}|g'(y)|t^2(1 + o(1))$.

5 Expansion for the transition densities

Our goal here is to obtain a second-order small-time approximation for the transition densities $\{p_t(\cdot; x)\}_{t \geq 0}$ of $\{X_t(x)\}_{t \geq 0}$. As it was done in the previous section, the idea is to work with the expansion (26) by first showing that each term there is differentiable with respect to y , and then determining their rates of convergence to 0 as $t \rightarrow 0$. One of the main difficulties of this approach comes from controlling the term corresponding to no “large” jumps. As in the case of purely diffusion processes, Malliavin calculus is proved to be the key tool for this task. This analysis is presented in the following subsection before our main result is presented in Section 5.2.

5.1 Density estimates for SDE with bounded jumps

In this part, we analyze the term corresponding to $N_t^\varepsilon = 0$:

$$\mathbb{P}(X_t(x) \geq x + y | N_t^\varepsilon = 0) = \mathbb{P}(X_t(\varepsilon, \emptyset, x) \geq x + y).$$

We will prove that, for any fixed positive integer N and $\eta > 0$, there exist an $\varepsilon_0 > 0$ and a constant $C < \infty$ (both only depending on N and η) such that the density $p_t(\cdot; \varepsilon, \emptyset, x)$ of $X_t(\varepsilon, \emptyset, x)$ satisfies

$$\sup_{|y-x| > \eta, \varepsilon < \varepsilon_0} p_t(y; \varepsilon, \emptyset, x) < Ct^N, \quad (30)$$

for all $0 < t \leq 1$. The estimate (30) holds true for a larger class of SDE with bounded jumps. Concretely, throughout this section, we consider the more general equation

$$X_t^x = x + \int_0^t b(X_{s-}^x) ds + \int_0^t \sigma(X_{s-}^x) dW_s + \int_0^t \int \gamma(X_{s-}^x, \zeta) \bar{M}(ds, d\zeta), \quad (31)$$

where, as before, $M(ds, d\zeta)$ is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R} \setminus \{0\}$ with mean measure $\mu(ds, d\zeta) = ds \times \nu(d\zeta)$ and $\bar{M} = M - \mu$ is its compensated measure. As before, we assume that ν admits a smooth density h with respect to the Lebesgue measure on $\mathbb{R} \setminus \{0\}$, but moreover, we now assume h is supported in a ball $B(0, r)$ for some $r \in (0, \infty)$. Note that the solution $(X_t(\varepsilon, \emptyset, x))_t$ of (16) perfectly fits within this framework.

The following assumptions on the coefficient functions of (31) are used in the sequel. Note that, when $\{X_t^x\}$ is taken to be $\{X_t(\varepsilon, \emptyset, x)\}$, these assumptions follow from the assumptions in Section 2.

Assumption 5.1 *The coefficients $b(x), \sigma(x)$ and $\gamma(x, \zeta)$ are 4 times differentiable. There exists a constant $c > 0$ and a function $\kappa \in \bigcap_{2 \leq p < \infty} \mathbb{L}^p(\nu)$ such that:*

1. $|\partial_{x^n} b(x)|, |\partial_{x^n} \sigma(x)|, \left| \frac{1}{\kappa(\zeta)} \partial_{x^n} \gamma(x, \zeta) \right| \leq c$, for $0 \leq n \leq 5$.
2. $\left| \partial_{x^n \zeta^k}^{n+k} \gamma(x, \zeta) \right| \leq c$, for $1 \leq n + k \leq 5$ and $k \geq 1$.

Malliavin calculus is the main tool to analyze the existence and smoothness of density for X_t^x . For the sake of completeness, we are presenting some necessary preliminaries of Malliavin calculus on Wiener-Poisson spaces in Appendix A following the approach in [7]. As described therein, there are different ways to define a Malliavin operator for such spaces. For our purposes, it suffices to consider the Malliavin operator corresponding to $\rho = 0$

(see Section A.2 for the details). The intuitive explanation of $\rho = 0$ is that when making perturbation of the sample path on the Wiener-Poisson space, we only perturb the Brownian path.

Let us start by noting that Assumption 5.1 ensures that $x \rightarrow X_t^x$ is a C^2 -diffeomorphism with a continuous density (see, e.g., [7]). Furthermore, recalling that H_∞ is the extended domain of the Malliavin operator, one has $X_t^x \in H_\infty$ and

$$U_t := \Gamma(X_t^x, X_t^x) = \left\{ \int_0^t \sigma^2(X_s^x) J_t(x)^{-2} ds \right\} J_t(x)^2. \quad (32)$$

Throughout this Section, we use the standard notation in which

$$J_t(x) = \frac{dX_t^x}{dx}. \quad (33)$$

Remark 5.2 Under the Condition (C4) of Section 2, $J_t(x)$ admits an inverse $Y_t := J_t(x)^{-1}$, almost surely. Indeed, one can show that (cf. [7])

$$\begin{aligned} dJ_t(x) &= 1 + \partial_x b(X_{t-}^x) J_{t-}(x) dt + \partial_x \sigma(X_{t-}^x) J_{t-}(x) dW_t \\ &\quad + \partial_x \gamma(X_{t-}^x, \zeta) J_{t-}(x) \bar{M}(dt, d\zeta), \end{aligned}$$

while $Y_t = J_t(x)^{-1}$ satisfies an equation of the form:

$$dY_t = 1 + Y_{t-} D_t dt + Y_{t-} E_t dW_t + Y_{t-} F_t \bar{M}(dt, d\zeta).$$

Here D_t, E_t and F_t are determined by $b(x), \sigma(x), \gamma(x, \zeta)$ and X_t^x . As a consequence, together with our assumption on b, σ and γ , one has

$$\mathbb{E} \sup_{0 \leq t \leq 1} J_t(x)^p, \text{ and } \mathbb{E} \sup_{0 \leq t \leq 1} J_t(x)^{-p} < \infty$$

for all $p > 1$.

The main result of this section is Theorem 5.7 below. For this purpose, we state some preliminary known results. Let us start with the following integration by parts formula (the main ingredient for existence and smoothness of the density of X_t^x), which is a special case of Lemma 4-14 in [7] together with the discussion of Chapter IV therein.

Proposition 5.3 (Integration by Parts) For any $f \in C_c^\infty(\mathbb{R})$, there exists a random variable $G_t \in L^p$ for all $p \in \mathbb{N}$, such that

$$\mathbb{E} \partial_x f(X_t^x) = \mathbb{E} G_t U_t^{-2} f(X_t^x).$$

The following existence and regularity result for the density of a finite measure is well known (see, e.g., Theorem 5.3 in [33]).

Proposition 5.4 Let m be a finite measure supported in an open set $O \subset \mathbb{R}^n$. Take any $p > n$. Suppose that there exists $g = (g_1, \dots, g_n) \in \mathbb{L}^p(m)$ such that

$$\int_{\mathbb{R}^n} \partial_j f dm = \int_{\mathbb{R}^n} f g_j dm, \quad f \in C_c^\infty(O), \quad j = 1, \dots, n.$$

Then m has a bounded density function $q \in C_b(O)$ satisfying

$$\|q\|_\infty \leq C \|g\|_{\mathbb{L}^p(m)}^n m(O)^{1-n/p}.$$

Here the constant C depends on n and p .

The following lemma is where we use assumption (C4). It is the first ingredient in proving Theorem 5.7.

Lemma 5.5 Recall $U_t = \Gamma(X_t, X_t)$. Under the Condition **(C4)** of Section 2, we have

$$\mathbb{E}U_t^{-p} \leq Ct^{-p},$$

for all $p > 1$.

Proof. The proof is a direct consequence of assumption **(C4)** and Remark 5.2. More precisely,

$$\begin{aligned} \mathbb{E}U_t^{-p} &= \mathbb{E} \frac{J_t(x)^{-2p}}{\left(\int_0^t J_s(x)^{-2} \sigma(X_s^x)^2 ds \right)^p} \leq \frac{1}{t^p} \mathbb{E} \frac{J_t(x)^{-2p}}{\delta^{2p} \inf_{0 \leq s \leq t} J_s(x)^{-2p}} \\ &= \frac{1}{t^p} \delta^{-2p} \mathbb{E} \left(J_t(x)^{-2p} \sup_{0 \leq s \leq 1} J_s(x)^{2p} \right). \end{aligned}$$

The proof is completed. ■

The last ingredient toward our main result of this section is the proposition below, which is just a restatement of Proposition 3.1.

Proposition 5.6 Let X_t be the solution to equation (31). Recall that since the Lévy density h is compactly supported and $\gamma(x, z)$ is uniformly bounded, the jumps of X_t are bounded by some constant S . There exists a constant $C(S)$ depending on S , such that, for any fixed $p > 0$ and all $0 \leq t \leq 1$,

$$\mathbb{P} \left\{ \sup_{0 \leq s \leq t} |X_s^x - x| \geq 2pS \right\} \leq C(S)t^p.$$

Finally, we can state and prove our main result of this section.

Theorem 5.7 Assume the Condition **(C3)** of Section 2 is satisfied. Let $\{X_t^x\}_{t \geq 0}$ be the solution to equation (31) and denote by $p_t(y; x)$ the density of X_t^x . Fix $\eta > 0$ and $N > 0$. Then, there exists $r(\eta, N) > 0$ such that if ν is supported in $B(0, r)$ with $r \leq r(\eta, N)$, we have for all $0 \leq t \leq 1$

$$\sup_{|x-y| \geq \eta} p_t(y; x) \leq C(\eta, N)t^N.$$

Proof. For a fix $t \geq 0$, define a finite measure m_t^η on \mathbb{R} by

$$m_t^\eta(A) = \mathbb{P} \left(\{X_t^x \in A \cap \bar{B}^c(x, \eta)\} \right), \quad A \subset \mathbb{R},$$

where $\bar{B}^c(x, r)$ denotes the complement of closure of $B(x, r)$. Thus, to prove our result it suffices to prove that m_t^η admits a density that has the desired bound. To this end, for any smooth function f compactly supported in $\bar{B}^c(x, \eta)$, we have:

$$\begin{aligned} \int_{\mathbb{R}} (\partial_x f)(y) m_t^\eta(dy) &= \mathbb{E} \partial_x f(X_t^x) = \mathbb{E} G_t U_t^{-2} f(X_t^x) \\ &= \int_{\mathbb{R}} \mathbb{E} [G_t U_t^{-2} | X_t^x = y] f(y) m_t^\eta(dy), \end{aligned}$$

where the second equality follows from integration by parts. The rest of the proof follows from Proposition 5.4, Lemma 5.5 and Proposition 5.6. ■

5.2 Expansion for the transition density

We are ready to state our main result of this section, namely, the second order expansion for the transition densities $\{p_t(\cdot; x)\}_{t \geq 0}$ of the process $\{X_t(x)\}_{t \geq 0}$ in terms of the Lévy density $g(x; y)$ defined in (27). The proof of the following result is presented in Appendix C.

Theorem 5.8 *Let $x \in \mathbb{R}$ and $y > 0$. Then, under the conditions and notation of Theorem 4.1, we have*

$$p_t(x+y; x) := -\frac{\partial \mathbb{P}(X_t(x) \geq x+y)}{\partial y} = ta_1(x; y) + \frac{t^2}{2}a_2(x; y) + O(t^3), \quad (34)$$

as $t \rightarrow 0$, where $a_1(x; y)$ and $a_2(x; y)$ admit the following representations (for $\varepsilon > 0$ small enough):

$$a_1(x; y) := g(x; y), \quad a_2(x; y) := \bar{\mathfrak{O}}(x; y) + \mathfrak{S}_1(x; y) + \mathfrak{S}_2(x; y),$$

with

$$\begin{aligned} \bar{\mathfrak{O}}(x; y) &= -\frac{\partial}{\partial y} \mathcal{D}(x; y), \\ \mathfrak{S}_1(x; y) &= \int (g(x; y - \gamma(x, \bar{\zeta})) + g(x; \bar{\gamma}(x+y, \bar{\zeta}) - x) \partial_u \bar{\gamma}(x+y, \bar{\zeta}) - 2g(x; y)) \bar{h}_\varepsilon(\bar{\zeta}) d\bar{\zeta}, \\ \mathfrak{S}_2(x; y) &= \int g(x + \gamma(x, \zeta); y - \gamma(x, \zeta)) h_\varepsilon(\zeta) d\zeta - 2g(x; y) \int h_\varepsilon(\zeta) d\zeta. \end{aligned} \quad (35)$$

6 The first order term of the option price expansion

In this section, we use our previous results to derive the leading term of the small-time expansion for option prices of out-of-the-money (OTM) European call options. This can be achieved by either the asymptotics of the tail distributions or the transition density. Given that the former requires less stringent conditions on the coefficients of the SDE, we choose the former approach.

It is well-known by practitioners that the market implied volatility skewness is more pronounced as the expiration time approaches. Such a phenomenon indicates that a jump risk should be included into classical purely-continuous financial models (e.g. local volatility models and stochastic volatility models) to reproduce more accurately the implied volatility skews observed in short-term option prices. Moreover, further studies have shown that accurate modeling of the option market and asset prices requires a mixture of a continuous diffusive component and a jump component (see [1], [2], [4], [29], [8], and [23]). The study of small-time asymptotics of option prices and implied volatilities has grown significantly during the last decade, as it provides a convenient tool for testing various pricing models and calibrating parameters in each model (see, e.g., [16], [10], [14], [6], [11], [31], [34], [15], [24], [13]). In spite of the ample literature on the asymptotic behavior of the transition densities and option prices for either purely-continuous or purely-jump models, results on local jump-diffusion models are scarce. Our result in this section is thus a first attempt in this direction.

Throughout this section, let $\{S_t\}_{t \geq 0}$ be the stock price process and let $X_t = \log S_t$ for each $t \geq 0$. We assume that \mathbb{P} is the option pricing measure and that under this measure the process $\{X_t\}_{t \geq 0}$ is of the form in (7). As usual, without loss of generality we assume that the risk-free interest rate r is 0. In particular, in order for $S_t = \exp X_t$ to be a \mathbb{Q} -(local) martingale, we fix

$$b(x) := -\frac{1}{2}\sigma^2(x) - \int (e^{\gamma(x, z)} - 1) h(z) dz$$

We assume that σ and γ are such that the Conditions **(C1-C4)** of Section 2 are satisfied. We also impose one extra condition **(C5)** for $h(z)$ and $\gamma(x, z)$ in order to derive option price expansion, as we are working with exponential of jump-dissusion now.

(C5) For each fixed $x \in \mathbb{R}$ there exists a constant $M(x) > 0$ such that $\int e^{3|\gamma(x, z)|} h(z) dz \leq M(x) < \infty$.

Note that this condition ensures immediately that $b(x)$ above is well defined.

By the Markov property of the system, it will suffice to compute a small-time expansion for

$$v_t = \mathbb{E}(S_t - K)_+ = \mathbb{E}(e^{X_t} - K)_+.$$

In particular, using the well-known formula

$$\mathbb{E}U\mathbf{1}_{\{U>K\}} = K\mathbb{P}\{U > K\} + \int_K^\infty \mathbb{P}\{U > s\}ds,$$

we can write

$$\mathbb{E}(e^{X_t} - K)_+ = \int_K^\infty \mathbb{P}\{S_t > s\}ds = S_0 \int_{\frac{K}{S_0}}^\infty \mathbb{P}\{X_t - x > \log s\}ds,$$

where $x = X_0 = \log S_0$. Recall that

$$\mathbb{P}(X_t - x \geq y) = e^{-\lambda_\varepsilon t} \sum_{n=0}^\infty \mathbb{P}(X_t - x \geq y | N_t^\varepsilon = n) \frac{(\lambda_\varepsilon t)^n}{n!}, \quad (36)$$

where $\lambda_\varepsilon := \int \phi_\varepsilon(\zeta)h(\zeta)d\zeta$ is the jump intensity of $\{N_t^\varepsilon\}_{t \geq 0}$. We proceed as in Section 4. First, note that

$$v_t = S_0 \int_{\frac{K}{S_0}}^\infty \mathbb{P}\{X_t - x > \log s\}ds = S_0 e^{-\lambda_\varepsilon t} (I_1 + I_2 + I_3), \quad (37)$$

where

$$\begin{aligned} I_1 &= \int_{\frac{K}{S_0}}^\infty \mathbb{P}\{X_t - x \geq \log s | N_t^\varepsilon = 0\}ds = \int_{\frac{K}{S_0}}^\infty \mathbb{P}\{X_t(\varepsilon, \emptyset, x) - x \geq \log s\}ds, \\ I_2 &= \lambda_\varepsilon t \int_{\frac{K}{S_0}}^\infty \mathbb{P}\{X_t - x \geq \log s | N_t^\varepsilon = 1\}ds, \\ I_3 &= \lambda_\varepsilon^2 t^2 \sum_{n=2}^\infty \frac{(\lambda_\varepsilon t)^{n-2}}{n!} \int_{\frac{K}{S_0}}^\infty \mathbb{P}\{X_t - x \geq \log s | N_t^\varepsilon = n\}ds. \end{aligned}$$

It is clear that $I_1/t \rightarrow 0$ as $t \rightarrow 0$ by Lemma 3.2. We show that the same is true for I_3 , which is the content of the following lemma. Its proof is given in Appendix D.

Lemma 6.1 *With the above notation, we have*

$$\sup_{n \in \mathbb{N}, t \in [0,1]} \frac{1}{n!} \int_0^\infty \mathbb{P}(|X_t - x| \geq \log y | N_t^\varepsilon = n) dy < \infty.$$

As a consequence, $I_3/t \rightarrow 0$ as $t \rightarrow 0$.

Note that the above lemma actually implies that $\mathbb{E}e^{|X_t - x|} < \infty$ for all $t \in [0,1]$. We are ready to state the main result of this section.

Theorem 6.2 *Let $v_t = \mathbb{E}(S_t - K)_+$ be the price of a European call option with strike K . Under the Conditions (C1-C5), we have*

$$\lim_{t \rightarrow 0} \frac{1}{t} v_t = \int_{-\infty}^\infty \left(S_0 e^{\gamma(x, \zeta)} - K \right)_+ h(\zeta) d\zeta.$$

Proof. We use the notation introduced in (37). Following a similar argument as in the proof of Lemma 6.1, one can show that

$$\int_{\frac{K}{S_0}}^\infty \sup_{t \in [0,1]} \mathbb{P}\{X_t - x \geq \log s | N_t^\varepsilon = 1\}ds < \infty. \quad (38)$$

Also, it is clear that I_1/t converges to 0 when t approaches to 0 by Lemma 3.2. Using the latter fact, (38), Lemma 6.1, (37), and dominated convergence theorem, we have

$$\lim_{t \rightarrow 0} \frac{v_t}{t} = \lim_{t \rightarrow 0} \frac{S_0 I_2}{t} = \lambda_\varepsilon S_0 \lim_{t \rightarrow 0} \int_{\frac{K}{S_0}}^\infty \mathbb{P}\{X_t - x \geq \log s | N_t^\varepsilon = 1\}ds = \lambda_\varepsilon S_0 \int_{\frac{K}{S_0}}^\infty \lim_{t \rightarrow 0} \mathbb{P}\{X_t - x \geq \log s | N_t^\varepsilon = 1\}ds.$$

Next, using Theorem 4.1, it follows that

$$\lim_{t \rightarrow 0} \frac{v_t}{t} = S_0 \int_{\frac{K}{S_0}}^{\infty} A_1(x, \log s) ds = S_0 \int_{\frac{K}{S_0}}^{\infty} \int_{\{\gamma(x, \zeta) \geq \log s\}} h(\zeta) d\zeta ds = \int_{-\infty}^{\infty} \left(S_0 e^{\gamma(x, \zeta)} - K \right)_+ h(\zeta) d\zeta,$$

where we had used Fubini's theorem for the last equality above. ■

Remark 6.3 As a special case of our result, let $\gamma(x, \zeta) = \zeta$. The model reduces to an exponential Lévy model. The above first order asymptotics becomes to

$$\lim_{t \rightarrow 0} \frac{1}{t} v_t = \int_{-\infty}^{\infty} (S_0 e^{\zeta} - K)_+ h(\zeta) d\zeta.$$

This recovers the well-known first-order asymptotic behavior for exponential Lévy model (see, e.g., [31] and [34]).

A Preliminaries of Malliavin calculus on Wiener-Poisson spaces

In this section, we follow [7] and introduce briefly the Malliavin calculus on Wiener-Poisson space to our need. All the details can be found in [7]. We first introduce, in an abstract manner, the notion of Malliavin operator (infinitesimal generator of the Ornstein-Uhlenbeck semigroup). Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $C_p^2(\mathbb{R}^n)$ be the space of all C^2 functions on \mathbb{R}^n which, together with all their partial derivatives up to the second order, have at most polynomial growth.

Definition A.1 A Malliavin operator (L, \mathcal{R}) is a linear operator L on a domain $\mathcal{R} \subset \bigcap_{p < \infty} \mathbb{L}^p$, taking values in $\bigcap_{p < \infty} \mathbb{L}^p$, and such that the following conditions hold true:

1. If $\Phi \in \mathcal{R}^n$ and $F \in C_p^2(\mathbb{R}^n)$, then $F(\Phi) \in \mathcal{R}$.
2. \mathcal{F} is the σ -field generated by all functions in \mathcal{R} .
3. L is symmetric in \mathbb{L}^2 , i.e. $\mathbb{E}(\Phi L \Psi) = \mathbb{E}(\Psi L \Phi)$ for all $\Phi, \Psi \in \mathcal{R}$.
4. Define

$$\Gamma(\Phi, \Psi) = L(\Phi \Psi) - \Phi L \Psi - \Psi L \Phi, \quad \text{for all } \Phi, \Psi \in \mathcal{R}.$$

Then Γ is nonnegative on $\mathcal{R} \times \mathcal{R}$.

5. For $\Phi = (\Phi^1, \dots, \Phi^n) \in \mathcal{R}^n$ and $F \in C_p^2(\mathbb{R}^n)$, one has

$$L(F \circ \Phi) = \sum_{i=1}^n \frac{\partial F}{\partial x_i}(\Phi) L \Phi^i + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 F}{\partial x_i \partial x_j}(\Phi) \Gamma(\Phi^i, \Phi^j).$$

A.1 Malliavin operator on Wiener-space

We consider the Wiener space of continuous paths:

$$\Omega^W = (\mathcal{C}([0, 1], \mathbb{R}), (\mathcal{F}_t^W)_{0 \leq t \leq 1}, \mathbb{P}^W)$$

where:

1. $\mathcal{C}([0, 1], \mathbb{R})$ is the space of continuous functions $[0, 1] \rightarrow \mathbb{R}$;
2. $(W_t)_{t \geq 0}$ is the coordinate process defined by $W_t(f) = f(t)$, $f \in \mathcal{C}([0, 1], \mathbb{R})$;
3. \mathbb{P}^W is the Wiener measure;

4. $(\mathcal{F}_t^W)_{0 \leq t \leq 1}$ is the $(\mathbb{P}^W\text{-completed})$ natural filtration of $(W_t)_{0 \leq t \leq 1}$.

Now, $(\Omega^W, \mathcal{F}^W, \mathbb{P}^W)$ is the canonical 1-dimensional Wiener space. Set

$$\mathcal{S} = \{\Phi = F(W_{t_1}, \dots, W_{t_n}), F \in C_p^2(\mathbb{R}^n), 0 \leq t_i \leq 1\}.$$

For Φ given as above, define

$$L^W \Phi = -\frac{1}{2} \sum_{i=1}^n \frac{\partial F}{\partial x_i}(W_{t_1}, \dots, W_{t_n}) W_{t_i} + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 F}{\partial x_i \partial x_j}(W_{t_1}, \dots, W_{t_n})(t_i \wedge t_j).$$

It is well-known that (L^W, \mathcal{S}^W) is a Malliavin operator. Furthermore, for $\Phi = F(W_{t_1}, \dots, W_{t_n})$ and $\Psi = H(W_{s_1}, \dots, W_{s_m})$

$$\Gamma^W(\Phi, \Psi) = \sum_{i=1}^n \sum_{j=1}^m \frac{\partial F}{\partial x_i}(W_{t_1}, \dots, W_{t_n}) \frac{\partial H}{\partial x_j}(W_{s_1}, \dots, W_{s_m})(t_i \wedge s_j).$$

A.2 Malliavin operator on Poisson space

In this section, we consider the canonical probability space $(\Omega^P, \mathcal{F}^P, \mathbb{P}^P)$ of a prefixed Poisson random measure $M(dt, d\zeta)$ on $[0, 1] \times (\mathbb{R} \setminus \{0\})$. Let $\mu(dt, d\zeta) = dt \times \nu(d\zeta)$ be the intensity measure of $M(dt, d\zeta)$ and denote by $\bar{M} = M - \mu$ its compensated Poisson random measure. Throughout our discussion, we also assume that ν admits a smooth density h with respect to the Lebesgue measure on \mathbb{R} , i.e. $\nu(d\zeta) = h(\zeta)d\zeta$.

We denote by $C_c^{b,2}([0, 1] \times \mathbb{R} \setminus \{0\})$ the set of all compactly supported, Borel functions f on $[0, 1] \times \mathbb{R} \setminus \{0\}$ that are of C^2 on $\mathbb{R} \setminus \{0\}$ with $f, \partial_\zeta f, \partial_\zeta^2 f$ uniformly bounded. For such f , we write

$$M(f) = \int f(t, \zeta) M(dt, d\zeta).$$

Let $\rho : \mathbb{R} \setminus \{0\} \rightarrow [0, \infty)$ be an auxiliary function which is of class C_b^1 . Set

$$\mathcal{S}^P = \{\Phi = F(M(f_1), \dots, M(f_k)); F \in C_p^2(\mathbb{R}^k), f_i \in C_c^{b,2}([0, 1] \times \mathbb{R} \setminus \{0\})\}.$$

Given any $\Phi \in \mathcal{S}^P$ as above, define

$$\begin{aligned} L^P \Phi &= \frac{1}{2} \sum_{i=1}^k \frac{\partial F}{\partial x_i}(M(f_1), \dots, M(f_k)) M \left(\rho \Delta_\zeta f_i + \partial_z \rho \partial_\zeta f_i + \rho \frac{\partial_\zeta h}{h} \partial_\zeta f_i \right) \\ &\quad + \frac{1}{2} \sum_{i,j=1}^k \frac{\partial^2 F}{\partial x_i \partial x_j}(M(f_1), \dots, M(f_k)) M(\rho \partial_\zeta f_i \partial_\zeta f_j). \end{aligned}$$

In the above, Δ_ζ stands for the Laplacian on $\mathbb{R} \setminus \{0\}$.

Proposition A.2 (L^P, \mathcal{S}^P) defines a Malliavin operator. Moreover for $\Phi(M(f_1), \dots, M(f_k))$ and $\Psi = H(M(h_1), \dots, M(h_l))$, we have

$$\Gamma^P(\Phi, \Psi) = \sum_{i=1}^k \sum_{j=1}^l \frac{\partial F}{\partial x_i}(M(f_1), \dots, M(f_k)) \frac{\partial H}{\partial x_j}(M(h_1), \dots, M(h_l)) M(\rho D_\zeta f_i D_\zeta h_j).$$

Proof. See Proposition 9-3 in [7]. ■

Remark A.3 As we can see from the above definition, the Malliavin operator for Poisson space is not uniquely defined. Indeed, each different choice of ρ gives a different operator. More remarks on this point are given at the end of this section below.

A.3 Malliavin operator on Wiener-Poisson space

Keep notations as in previous sections. We define a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega^W, \mathcal{F}^W, \mathbb{P}^W) \otimes (\Omega^P, \mathcal{F}^P, \mathbb{P}^P).$$

Clearly $(\Omega, \mathcal{F}, \mathbb{P})$ is the canonical Wiener-Poisson space. Then define (L, \mathcal{S}) to be the direct product of (L^W, \mathcal{S}^W) and (L^P, \mathcal{S}^P) . More precisely,

1. $\mathcal{S} = \{\Phi = F(\Phi_1, \dots, \Phi_n; \Psi_1, \dots, \Psi_m); \text{ for all } \Phi_i \in \mathcal{S}^W, \Psi_j \in \mathcal{S}^P, F \in C_p^2\}$.
2. For Φ as given in (1), and $\omega = (\omega^W, \omega^P) \in \Omega$,

$$L\Phi(\omega) = L^W\Phi(\cdot, \omega^P) + L^P\Phi(\omega^W, \cdot).$$

It can be shown that (L, \mathcal{S}) is a Malliavin operator on $(\Omega, \mathcal{F}, \mathbb{P})$. A few properties of (L, \mathcal{S}) are listed below.

Proposition A.4 *Let $\Phi \in \mathcal{S}^W$ and $\Psi \in \mathcal{S}^P$, we have*

1. $L\Phi = L^W\Phi$, and $\Gamma(\Phi, \Phi) = \Gamma^W(\Phi, \Phi)$.
2. $L\Psi = L^P\Psi$, and $\Gamma(\Psi, \Psi) = \Gamma^P(\Psi, \Psi)$.
3. $\Gamma(\Phi, \Psi) = 0$.

Finally, we extend the L to a larger domain that is suitable to work on. For this purpose, set

$$\|\Phi\|_{H_p} = \|\Phi\|_{\mathbb{L}^p} + \|L\Phi\|_{\mathbb{L}^p} + \|\Gamma(\Phi, \Phi)^{1/2}\|_{\mathbb{L}^p}$$

and denote by H_p the closure of \mathcal{S} under $\|\cdot\|_{H_p}$. Also set

$$H_\infty = \bigcap_{2 \leq p < \infty} H_p.$$

Proposition A.5 *The operator (L, H_∞) is a Malliavin operator.*

Proof. This is the content of Section 8-b in [7]. ■

Remark A.6 *As before, a different choice of the auxiliary function ρ in the definition for L^P results in a different Malliavin operator L on the Wiener-Poisson space. Hence, the extended domain H_∞ may also be different.*

Remark A.7 *Consider the following stochastic differential equation with regular enough coefficients*

$$X_t^x = x + \int_0^t b(X_{s-}^x) ds + \int_0^t \sigma(X_{s-}^x) dW_s + \int_0^t \int \gamma(X_{s-}^x, \zeta) \bar{M}(ds, d\zeta).$$

To make use of the power of Malliavin calculus to analyze the above SDE, one needs to impose further conditions on ρ so that X_t^x is in the domain H_∞ . Those further conditions are generally determined by the support of the intensity measure ν and are needed to ensure that some approximation procedure can work out. For more details of the role that ρ plays, see Section 9-1 and Theorem 10-3 in [7]. It is important to note that, among various choices of ρ , one can always choose $\rho = 0$, as we impose the non-degeneracy Assumption **(C3)**. The intuitive explanation of $\rho = 0$ is that when making perturbation of the sample path on the Wiener-Poisson space, we only perturb the Brownian path.

B Proof of the tail distribution expansion

The proof of Theorem 4.1 is decomposed into three steps described in the following three subsections. For future use in obtaining the expansion for the transition densities, we will write explicitly the reminder terms when applying Dynkin's formula (23) or in any other type of approximation.

B.1 Key lemma to control the tail of the process with one large jump

The following result will allow us to obtain the second-order expansion for the process with one large jump.

Lemma B.1 *Under the setting and conditions (C1-C4) of Section 2,*

$$\mathbb{P}(X_t(\varepsilon, \emptyset, z + \gamma(z, J)) \geq \vartheta) = H_0(z; \vartheta) + tH_1(z; \vartheta) + t^2\check{\mathcal{R}}_t(z; \vartheta), \quad (39)$$

for any $z, \vartheta \in \mathbb{R}$, where

$$\begin{aligned} H_0(z; \vartheta) &:= \mathbb{P}(\gamma(z, J) + z \geq \vartheta), & H_1(z; \vartheta) &:= D(z; \vartheta) + I(z; \vartheta), \\ D(z; \vartheta) &:= \tilde{\Gamma}(\vartheta; z)b(\vartheta) - \partial_\vartheta \tilde{\Gamma}(\vartheta; z)v(\vartheta) - \tilde{\Gamma}(\vartheta; z)v'(\vartheta), \\ I(z; \vartheta) &:= \int [\mathbb{P}(z + \gamma(z, J) \geq \bar{\gamma}(\vartheta, \zeta)) - \mathbb{P}(z + \gamma(z, J) \geq \vartheta)] \bar{h}_\varepsilon(\zeta) d\zeta, \end{aligned}$$

and, for $\varepsilon > 0$ small enough,

$$\limsup_{t \rightarrow 0} \sup_{z \in \mathbb{R}} |\check{\mathcal{R}}_t^1(z; \vartheta)| < \infty, \quad \sup_{z, \vartheta} |H_1(z; \vartheta)| < \infty.$$

The idea to obtain (39) consists of approximating the function $\mathbf{1}_{\{X_t(\varepsilon, \emptyset, z + \gamma(z, J)) \geq \vartheta\}}$ by a smooth sequence of functions $f_\delta(X_t(\varepsilon, \emptyset, z + \gamma(z, J)))$, $\delta > 0$. Concretely, we let

$$f_\delta(w) := \int_{-\infty}^{w-\vartheta} \varphi_\delta(u) du,$$

where $\varphi_\delta(x) := \delta^{-1}\varphi(\delta^{-1}x)$ for a density function $\varphi \in C^\infty$ with $\text{supp } \varphi \subset [-1, 1]$. In particular, as $\delta \rightarrow 0$,

$$f_\delta(w) \rightarrow \mathbf{1}_{\{w \geq \vartheta\}}, \quad \text{and} \quad \int g(w) f'_\delta(w) dw = \int g(w) \varphi_\delta(w - \vartheta) dw \rightarrow g(\vartheta), \quad (40)$$

whenever g is bounded and continuous at ϑ . It is then natural to apply Dynkin's formula to $f_\delta(X_t(\varepsilon, \emptyset, z + \gamma(z, J)))$ and show that each of the resulting terms is convergent when $\delta \rightarrow 0$. The following result, whose proof is presented in Appendix D, is needed to formalize the last step.

Lemma B.2 *Let $\tilde{\Gamma}(\cdot; z)$ be the density of the random variable $z + \gamma(z, J)$ and let $p_t(\cdot; \varepsilon, \emptyset, \zeta)$ be the density of $X_t(\varepsilon, \emptyset, \zeta)$. Then, under the Conditions (C1-C4) of Section 2, there exists an $\varepsilon > 0$ small enough such that for any compact set $K \subset \mathbb{R}$,*

$$\limsup_{t \rightarrow 0} \sup_{z \in \mathbb{R}} \sup_{\eta \in K} \left| \frac{\partial^k}{\partial \eta^k} \int \tilde{\Gamma}(\zeta; z) p_t(\eta; \varepsilon, \emptyset, \zeta) d\zeta \right| < \infty, \quad k \geq 0. \quad (41)$$

We are now in position to show (39). **Proof of Lemma B.1.** Throughout, $\partial_y \gamma$ and $\partial_\zeta \gamma$ will denote the partial derivatives of $\gamma(y, \zeta)$ with respect to its first and second arguments, respectively. By dominated convergence theorem, we have

$$\mathbb{P}(X_t(\varepsilon, \emptyset, z + \gamma(z, J)) \geq \vartheta) = \lim_{\delta \downarrow 0} \mathbb{E} f_\delta(X_t(\varepsilon, \emptyset, z + \gamma(z, J))). \quad (42)$$

Note that

$$\mathbb{E} f_\delta(X_t(\varepsilon, \emptyset, z + \gamma(z, J))) = \int \tilde{\Gamma}(\zeta; z) \mathbb{E} f_\delta(X_t(\varepsilon, \emptyset, \zeta)) d\zeta, \quad (43)$$

and, thus, applying Dynkin's formula (24) to the expectation in the above integral gives us

$$\mathbb{E}f_\delta(X_t(\varepsilon, \emptyset, z + \gamma(z, J))) = \int \tilde{\Gamma}(\zeta; z) f_\delta(\zeta) d\zeta + t \int \tilde{\Gamma}(\zeta; z) L_\varepsilon f_\delta(\zeta) d\zeta \quad (44)$$

$$+ t^2 \int \tilde{\Gamma}(\zeta; z) \int_0^1 (1 - \alpha) \mathbb{E}(L_\varepsilon)^2 f_\delta(X_{\alpha t}(\varepsilon, \emptyset, \zeta)) d\alpha d\zeta. \quad (45)$$

We analyze the limit of each of the three terms on the right-hand side of the previous equation. By dominated convergence theorem, the leading term of (42) is given by

$$H_0(z; \vartheta) := \lim_{\delta \downarrow 0} \int \tilde{\Gamma}(\zeta; z) f_\delta(\zeta) d\zeta = \int \tilde{\Gamma}(\zeta; z) I_{[\vartheta, \infty)}(\zeta) d\zeta = \mathbb{P}\{\gamma(z, J) + z \geq \vartheta\}.$$

To compute the limit of the subsequent term, recall $L_\varepsilon f_\delta = \mathcal{D}_\varepsilon f_\delta + \mathcal{I}_\varepsilon f_\delta$ with \mathcal{D}_ε and \mathcal{I}_ε defined as in (20). Then, the term of order t has two contribution:

$$A_\delta := \int \tilde{\Gamma}(\zeta; z) \mathcal{D}_\varepsilon f_\delta(\zeta) d\zeta, \quad B_\delta := \int \tilde{\Gamma}(\zeta; z) \mathcal{I}_\varepsilon f_\delta(\zeta) d\zeta.$$

Using that $f'_\delta(\zeta) = \varphi_\delta(\zeta - \vartheta)$ and by integration by parts, it follows that

$$A_\delta = \int \left(\tilde{\Gamma}(\zeta; z) b(\zeta) - \partial_\zeta \tilde{\Gamma}(\zeta; z) v(\zeta) - \tilde{\Gamma}(\zeta; z) v'(\zeta) \right) \varphi_\delta(\zeta - \vartheta) d\zeta.$$

Applying (40) and Lemma 2.3-(2),

$$\lim_{\delta \downarrow 0} A_\delta = \tilde{\Gamma}(\vartheta; z) b(\vartheta) - \partial_\vartheta \tilde{\Gamma}(\vartheta; z) v(\vartheta) - \tilde{\Gamma}(\vartheta; z) v'(\vartheta).$$

We now analyze the limit of the second term B_δ . Applying (91) given in the Appendix D below, we can write B_δ as

$$B_\delta = \int f'_\delta(w) \int \int_{\bar{\gamma}(w, \zeta)}^w \tilde{\Gamma}(\eta; z) d\eta \bar{h}_\varepsilon(\zeta) d\zeta dw = \int \varphi_\delta(w - \vartheta) \int \int_{\bar{\gamma}(w, \zeta)}^w \tilde{\Gamma}(\eta; z) d\eta \bar{h}_\varepsilon(\zeta) d\zeta dw. \quad (46)$$

Since

$$\begin{aligned} \int \int_{\bar{\gamma}(w, \zeta)}^w \tilde{\Gamma}(\eta; z) d\eta \bar{h}_\varepsilon(\zeta) d\zeta &= - \int \int_0^1 \tilde{\Gamma}(\bar{\gamma}(w, \zeta\beta); z) (\partial_\zeta \bar{\gamma})(w, \zeta\beta) d\beta \bar{h}_\varepsilon(\zeta) \zeta d\zeta \\ &= \int \int_0^1 \tilde{\Gamma}(\bar{\gamma}(w, \zeta\beta); z) \frac{(\partial_\zeta \gamma)(\bar{\gamma}(w, \zeta\beta), \zeta\beta)}{1 + (\partial_y \gamma)(\bar{\gamma}(w, \zeta\beta), \zeta\beta)} d\beta \bar{h}_\varepsilon(\zeta) \zeta d\zeta, \end{aligned}$$

where we used (80) in the second equality, conditions **(C2)** and **(C4)** imply that the factor multiplying $\varphi_\delta(w - \vartheta)$ in (46) is bounded and continuous in w and, thus,

$$\lim_{\delta \downarrow 0} B_\delta = \int \int_{\bar{\gamma}(\vartheta, \zeta)}^\vartheta \tilde{\Gamma}(\eta; z) d\eta \bar{h}_\varepsilon(\zeta) d\zeta =: B_0(z; \vartheta).$$

Next, recalling that $\tilde{\Gamma}(\zeta; z)$ is the density of $\tilde{J} := z + \gamma(z, J)$,

$$B_0(z; \vartheta) = \int (\mathbb{P}(z + \gamma(z, J) \geq \bar{\gamma}(\vartheta, \zeta)) - \mathbb{P}(z + \gamma(z, J) \geq \vartheta)) \bar{h}_\varepsilon(\zeta) d\zeta.$$

Putting together the previous two limits, we obtain the term of order t :

$$H_1(z; \vartheta) := \lim_{\delta \downarrow 0} \int \tilde{\Gamma}(\zeta; z) L_\varepsilon f_\delta(\zeta) d\zeta = D(z; \vartheta) + I(z; \vartheta),$$

with $D(z; \vartheta)$ and $I(z; \vartheta)$ given as in the statement of Lemma.

Finally, we estimate the remainder term

$$\check{\mathcal{R}}_t(z; \vartheta) := \lim_{\delta \downarrow 0} \int \tilde{\Gamma}(\zeta; z) \int_0^1 (1 - \alpha) \mathbb{E}(L_\varepsilon)^2 f_\delta(X_{\alpha t}(\varepsilon, \emptyset, \zeta)) d\alpha d\zeta \quad (47)$$

and show that this is uniformly bounded for t small enough. Let $\check{\mathcal{R}}_t(z; \vartheta; \delta, \varepsilon)$ be the expression following $\lim_{\delta \downarrow 0}$ and note that

$$\begin{aligned} \check{\mathcal{R}}_t(z; \vartheta, \delta, \varepsilon) &= \int \tilde{\Gamma}(\zeta; z) \int_0^1 (1 - \alpha) \mathbb{E}(\mathcal{D}_\varepsilon)^2 f_\delta(X_{\alpha t}(\varepsilon, \emptyset, \zeta)) d\alpha d\zeta \\ &\quad + \int \tilde{\Gamma}(\zeta; z) \int_0^1 (1 - \alpha) \mathbb{E}(\mathcal{I}_\varepsilon)^2 f_\delta(X_{\alpha t}(\varepsilon, \emptyset, \zeta)) d\alpha d\zeta \\ &\quad + \int \tilde{\Gamma}(\zeta; z) \int_0^1 (1 - \alpha) \mathbb{E} \mathcal{I}_\varepsilon \mathcal{D}_\varepsilon f_\delta(X_{\alpha t}(\varepsilon, \emptyset, \zeta)) d\alpha d\zeta \\ &\quad + \int \tilde{\Gamma}(\zeta; z) d\zeta \int_0^1 (1 - \alpha) \mathbb{E} \mathcal{D}_\varepsilon \mathcal{I}_\varepsilon f_\delta(X_{\alpha t}(\varepsilon, \emptyset, \zeta)) d\alpha d\zeta. \end{aligned} \quad (48)$$

We shall use Lemmas B.2 and D.2 to deal with the four terms on the right-hand side of the previous equation. For simplicity, we only give the details for second term, that we denote hereafter $\bar{I}_t^{(2)}(\vartheta; \delta, \varepsilon, z)$. The other terms can similarly be handled. Using (90)-(91) and Fubini,

$$\begin{aligned} \bar{I}_t^{(2)}(z; \vartheta, \delta, \varepsilon) &= \int \tilde{\Gamma}(\zeta; z) \int_0^1 (1 - \alpha) \int \mathcal{I}_\varepsilon^2 f_\delta(w) p_{\alpha t}(w; \varepsilon, \emptyset, \zeta) dw d\alpha d\zeta \\ &= \int \tilde{\Gamma}(\zeta; z) \int_0^1 (1 - \alpha) \int \mathcal{I}_\varepsilon f_\delta(w) \tilde{\mathcal{I}}_\varepsilon p_{\alpha t}(w; \varepsilon, \emptyset, \zeta) dw d\alpha d\zeta \\ &= \int \tilde{\Gamma}(\zeta; z) \int_0^1 (1 - \alpha) \int f'_\delta(w) \int \int_{\bar{\gamma}(w, \tilde{\zeta})}^w \tilde{\mathcal{I}}_\varepsilon p_{\alpha t}(\eta; \varepsilon, \emptyset, \zeta) d\eta \bar{h}_\varepsilon(\tilde{\zeta}) d\tilde{\zeta} dw d\alpha d\zeta \\ &= \int f'_\delta(w) \int_0^1 (1 - \alpha) \int \int_{\bar{\gamma}(w, \tilde{\zeta})}^w \tilde{\Gamma}(\zeta; z) \tilde{\mathcal{I}}_\varepsilon p_{\alpha t}(\eta; \varepsilon, \emptyset, \zeta) d\zeta d\eta \bar{h}_\varepsilon(\tilde{\zeta}) d\tilde{\zeta} d\alpha dw. \end{aligned}$$

Using Fubini, it follows that

$$\int \tilde{\Gamma}(\zeta; z) \tilde{\mathcal{I}}_\varepsilon p_{\alpha t}(\eta; \varepsilon, \emptyset, \zeta) d\zeta = \tilde{\mathcal{I}}_\varepsilon \check{p}_{\alpha t}(\eta; z, \varepsilon), \quad \text{where} \quad \check{p}_t(\eta; z, \varepsilon) := \int \tilde{\Gamma}(\zeta; z) p_t(\eta; \varepsilon, \emptyset, \zeta) d\zeta.$$

Then, using the identity

$$\begin{aligned} \tilde{\mathcal{I}}_\varepsilon g(y) &= - \int \int_0^1 g'(\bar{\gamma}(y, \zeta \beta)) (\partial_\zeta \bar{\gamma})(\eta, \zeta \beta) d\beta \frac{\zeta \bar{h}_\varepsilon(\zeta)}{1 + (\partial_y \gamma)(\bar{\gamma}(y, \zeta), \zeta)} d\zeta \\ &\quad - g(y) \int (\partial_y \gamma)(\bar{\gamma}(y, \zeta), \zeta) \frac{\bar{h}_\varepsilon(\zeta)}{1 + (\partial_y \gamma)(\bar{\gamma}(y, \zeta), \zeta)} d\zeta, \end{aligned}$$

we have

$$\begin{aligned} \bar{I}_t^{(2)}(z; \vartheta, \delta, \varepsilon) &= \int f'_\delta(w) \int_0^1 (1 - \alpha) \int \int_{\bar{\gamma}(w, \tilde{\zeta})}^w \tilde{\mathcal{I}}_\varepsilon \check{p}_{\alpha t}(\eta; z, \varepsilon) d\eta \bar{h}_\varepsilon(\tilde{\zeta}) d\tilde{\zeta} d\alpha dw \\ &= \int f'_\delta(w) \bar{I}_t^{(2,1)}(w; z, \varepsilon) dw + \int f'_\delta(w) \bar{I}_t^{(2,2)}(w; z, \varepsilon) dw, \end{aligned}$$

where

$$\begin{aligned}\bar{I}_t^{(2,1)}(w; z, \varepsilon) &:= - \int_0^1 (1 - \alpha) \int \int_{\bar{\gamma}(w, \tilde{\zeta})}^w \check{p}_{\alpha t}(\eta; z, \varepsilon) \int \frac{(\partial_y \gamma)(\bar{\gamma}(\eta, \tilde{\zeta}), \tilde{\zeta}) \bar{h}_\varepsilon(\tilde{\zeta})}{1 + (\partial_y \gamma)(\bar{\gamma}(\eta, \tilde{\zeta}), \tilde{\zeta})} d\tilde{\zeta} d\eta \bar{h}_\varepsilon(\tilde{\zeta}) d\tilde{\zeta} d\alpha \\ \bar{I}_t^{(2,2)}(w; z, \varepsilon) &:= - \int_0^1 (1 - \alpha) \int \int_{\bar{\gamma}(w, \tilde{\zeta})}^w \int \int_0^1 \check{p}'_{\alpha t}(\bar{\gamma}(\eta, \tilde{\zeta}\beta); z, \varepsilon) (\partial_\zeta \bar{\gamma})(\eta, \tilde{\zeta}\beta) d\beta \\ &\quad \times \frac{\tilde{\zeta} \bar{h}_\varepsilon(\tilde{\zeta})}{1 + (\partial_y \gamma)(\bar{\gamma}(\eta, \tilde{\zeta}), \tilde{\zeta})} d\tilde{\zeta} d\eta \bar{h}_\varepsilon(\tilde{\zeta}) d\tilde{\zeta} d\alpha.\end{aligned}\tag{49}$$

We next show that each $\bar{I}_t^{(2,i)}(w; z, \varepsilon)$ is uniformly bounded in w and z for t small enough. Indeed, using (10) and (11-i),

$$\begin{aligned}|\bar{I}_t^{(2,1)}(w; z, \varepsilon)| &\leq C \int_0^1 (1 - \alpha) \int \int_{\bar{\gamma}(w, \tilde{\zeta}) \wedge w}^{\bar{\gamma}(w, \tilde{\zeta}) \vee w} \check{p}_{\alpha t}(\eta; z, \varepsilon) d\eta \bar{h}_\varepsilon(\tilde{\zeta}) d\tilde{\zeta} d\alpha \\ &\leq C \int_0^1 (1 - \alpha) \int \int_0^1 \check{p}_{\alpha t}(\bar{\gamma}(w, \tilde{\zeta}\beta); z, \varepsilon) |(\partial_\zeta \bar{\gamma})(w, \tilde{\zeta}\beta)| d\beta \tilde{\zeta} \bar{h}_\varepsilon(\tilde{\zeta}) d\tilde{\zeta} d\alpha \\ &\leq C \int \int_0^1 \int_0^1 (1 - \alpha) \check{p}_{\alpha t}(\bar{\gamma}(w, \tilde{\zeta}\beta); z, \varepsilon) d\alpha d\beta \tilde{\zeta} \bar{h}_\varepsilon(\tilde{\zeta}) d\tilde{\zeta},\end{aligned}$$

for some constants C (that may change from line to line). Using (41) with $k = 0$, it follows that, for $\varepsilon, t > 0$ small enough,

$$\sup_{z \in \mathbb{R}, w \in \text{supp} f_1} |\bar{I}_t^{(2,1)}(w; z, \varepsilon)| < \infty.\tag{50}$$

Similarly,

$$\begin{aligned}|\bar{I}_t^{(2,2)}(w; z, \varepsilon)| &\leq C \int_0^1 (1 - \alpha) \int \int_{\bar{\gamma}(w, \tilde{\zeta}) \wedge w}^{\bar{\gamma}(w, \tilde{\zeta}) \vee w} \int \int_0^1 |\check{p}'_{\alpha t}(\bar{\gamma}(\eta, \tilde{\zeta}\beta); z, \varepsilon)| d\beta \tilde{\zeta} \bar{h}_\varepsilon(\tilde{\zeta}) d\tilde{\zeta} d\eta \bar{h}_\varepsilon(\tilde{\zeta}) d\tilde{\zeta} d\alpha \\ &\leq C \int_0^1 (1 - \alpha) \int \int_0^1 \int \int_0^1 |\check{p}'_{\alpha t}(\bar{\gamma}(\eta, \tilde{\zeta}\beta); z, \varepsilon)| d\beta \tilde{\zeta} \bar{h}_\varepsilon(\tilde{\zeta}) d\tilde{\zeta} d\beta \tilde{\zeta} \bar{h}_\varepsilon(\tilde{\zeta}) d\tilde{\zeta} d\alpha,\end{aligned}$$

which, again using (41) with $k = 1$, leads us to

$$\sup_{z \in \mathbb{R}, w \in \text{supp} f_1} |\bar{I}_t^{(2,2)}(w; z, \varepsilon)| < \infty,\tag{51}$$

for $\varepsilon, t > 0$ small enough. Due to (50-51) and the continuity of the functions $\bar{I}_2^{(i)}(\delta; w; z, t)$ in w ,

$$\lim_{\delta \downarrow 0} \bar{I}_t^{(2)}(z; \vartheta; \delta, \varepsilon) = \bar{I}_t^{(2,1)}(\vartheta; z, \varepsilon) + \bar{I}_t^{(2,2)}(\vartheta; z, \varepsilon),\tag{52}$$

which is uniformly bounded in z for any fixed ϑ and $0 < t < t_0$ with $t_0 > 0$ small enough. ■

B.2 The leading term

In order to determine the leading term of (25), we analyze the second term in (26) corresponding to $n = 1$ (only one “large” jump). By conditioning on the time of the jump (necessarily uniformly distributed on $[0, t]$),

$$\mathbb{P}(X_t(x) \geq x + y | N_t^\varepsilon = 1) = \frac{1}{t} \int_0^t \mathbb{P}(X_t(\varepsilon, \{s\}, x) \geq x + y) ds.\tag{53}$$

Conditioning on \mathcal{F}_{s-} ,

$$\mathbb{P}(X_t(\varepsilon, \{s\}, x) \geq x + y) = \mathbb{E}(G_{t-s}(X_{s-}(\varepsilon, \emptyset, x))) = \mathbb{E}(G_{t-s}(X_s(\varepsilon, \emptyset, x))),\tag{54}$$

where

$$G_t(z) := G_t(z; x, y) := \mathbb{P}[X_t(\varepsilon, \emptyset, z + \gamma(z, J)) \geq x + y]. \quad (55)$$

Using Lemma B.1,

$$\begin{aligned} \mathbb{P}(X_t(\varepsilon, \{s\}, x) \geq x + y) &= \mathbb{E}H_0(X_s(\varepsilon, \emptyset, x); x + y) + (t - s)\mathbb{E}H_1(X_s(\varepsilon, \emptyset, x); x + y) \\ &\quad + (t - s)^2\mathbb{E}\mathcal{R}_{t-s}^1(X_s(\varepsilon, \emptyset, x); x, y), \end{aligned} \quad (56)$$

where $\mathcal{R}_t^1(w; x, y) := \check{\mathcal{R}}_t(w; x + y)$. Next, we apply the Dynkin's formula (23) to $\mathbb{E}H_0(X_s(\varepsilon, \emptyset, x); x + y)$, which is valid since $H_0(z; x + y) = \mathbb{P}(\gamma(z, J) + z \geq x + y)$ is C_b^4 in light of Lemma 2.3-(3). By (23),

$$\mathbb{E}H_0(X_s(\varepsilon, \emptyset, x); x + y) = H_{0,0}(x; y) + sH_{0,1}(x; y) + s^2\mathcal{R}_s^2(x; y), \quad (57)$$

where

$$\begin{aligned} H_{0,0}(x; y) &:= H_0(x; x + y) = \mathbb{P}[\gamma(x, J) \geq y], \\ H_{0,1}(x; y) &:= L_\varepsilon H_0(x; x + y) = b(x) \frac{\partial H_0(z; x + y)}{\partial z} \Big|_{z=x} + \frac{\sigma^2(x)}{2} \frac{\partial^2 H_0(z; x + y)}{\partial z^2} \Big|_{z=x} \\ &\quad + \int (H_0(x + \gamma(x, \zeta); x + y) - H_0(x; x + y)) \bar{h}_\varepsilon(\zeta) d\zeta \\ &= b(x) \frac{\partial \mathbb{P}[\tilde{\gamma}(z, J) \geq x + y]}{\partial z} \Big|_{z=x} + \frac{\sigma^2(x)}{2} \frac{\partial^2 \mathbb{P}[\tilde{\gamma}(z, J) \geq x + y]}{\partial z^2} \Big|_{z=x} + \hat{H}_{0,1}(x; y), \\ \mathcal{R}_s^2(x; y) &:= \int_0^1 (1 - \alpha) \mathbb{E}L_\varepsilon^2 H_0(X_{\alpha s}(\varepsilon, \emptyset, x); x + y) d\alpha, \end{aligned} \quad (58)$$

with $\hat{H}_{0,1}(x; y)$ given by

$$\hat{H}_{0,1}(x; y) = \int (\mathbb{P}[\gamma(x + \gamma(x, \zeta), J) \geq y - \gamma(x, \zeta)] - \mathbb{P}[\gamma(x, J) \geq y]) \bar{h}_\varepsilon(\zeta) d\zeta. \quad (59)$$

Note that $\sup_{s < 1, x, y} |\mathcal{R}_s^2(x; y)| < \infty$ in light of (24) and, also, by writing $\mathbb{P}[\tilde{\gamma}(z, J) \geq x + y] = \mathbb{P}[\gamma(z, J) \geq x + y - z] = G(z, x + y - z)$ with $G(x, y) = \mathbb{P}(\gamma(x, J) \geq y)$, we get

$$\frac{\partial \mathbb{P}[\tilde{\gamma}(z, J) \geq x + y]}{\partial z} \Big|_{z=x} = \frac{\partial \mathbb{P}[\gamma(x, J) \geq y]}{\partial x} + \Gamma(y; x), \quad (60)$$

$$\frac{\partial^2 \mathbb{P}[\tilde{\gamma}(z, J) \geq x + y]}{\partial z^2} \Big|_{z=x} = \frac{\partial^2 \mathbb{P}[\gamma(x, J) \geq y]}{\partial x^2} + 2 \frac{\partial \Gamma(y; x)}{\partial x} - \frac{\partial \Gamma(y; x)}{\partial y}. \quad (61)$$

Plugging (57) in (56) and recalling from Lemma B.1 that the second and third terms on the right hand side of (56) are bounded for t small enough, $\mathbb{P}(X_t(\varepsilon, \{s\}, x) \geq x + y) = \mathbb{P}[\gamma(x, J) \geq y] + O(t)$, which can then be plugged in (53) to get

$$\mathbb{P}(X_t(x) \geq x + y | N_t^\varepsilon = 1) = \mathbb{P}[\gamma(x, J) \geq y] + O(t).$$

Finally, (26) can be written as

$$\mathbb{P}(X_t(x) \geq x + y) = e^{-\lambda_\varepsilon t} t \lambda_\varepsilon \mathbb{P}[\gamma(x, J) \geq y] + O(t^2) = t \int \mathbf{1}_{\{\gamma(x, \zeta) \geq y\}} h(\zeta) d\zeta + O(t^2),$$

where the second equality above is valid for $\varepsilon > 0$ small enough.

B.3 Second order term

In addition to (57), we also need to consider the leading terms in the term $\mathbb{E}H_1(X_s(\varepsilon, \emptyset, x); x+y)$ of (56) and the term $\mathbb{P}(X_t(x) \geq x+y | N_t^\varepsilon = 2)$ of (26). Let us first show that $z \rightarrow H_1(z; x+y)$ is C_b^2 . To this end, let

$$\mathcal{K}(\zeta; x, y, z) := \mathbb{P}[z + \gamma(z, J) \geq \bar{\gamma}(x+y, \zeta)] - \mathbb{P}[z + \gamma(z, J) \geq x+y],$$

and recall that

$$H_1(z; x+y) = \tilde{\Gamma}(x+y; z)b(x+y) - (\partial_\zeta \tilde{\Gamma})(x+y; z)v(x+y) - \tilde{\Gamma}(x+y; z)v'(x+y) + \int \mathcal{K}(\zeta; x, y, z)\bar{h}_\varepsilon(\zeta)d\zeta,$$

where $\partial_\zeta \tilde{\Gamma}$ and $\partial_z \tilde{\Gamma}$ denote the partial derivatives of the density $\tilde{\Gamma}(\zeta; z)$. Obviously, the first three terms on the right-hand side of the previous expression are C_b^2 in light of Lemma 2.3-(2). Hence, for the derivative $\partial_z H_1(z; x+y)$ to exist, it suffices to show that $\partial_z \mathcal{K}(\zeta; x, y, z)$ exists and that

$$\sup_{z, x, y} \left| \frac{\partial \mathcal{K}(\zeta; x, y, z)}{\partial z} \right| < C|\zeta|, \quad (62)$$

for any $|\zeta| < \varepsilon$ and some constant $C < \infty$. Writing $\mathcal{K}(\zeta; x, y, z) = \int_{\bar{\gamma}(x+y, \zeta)}^{x+y} \tilde{\Gamma}(\eta; z)d\eta$ and using that $\tilde{\Gamma}(\eta; z) \in C_b^\infty$, we have

$$\frac{\partial \mathcal{K}(\zeta; x, y, z)}{\partial z} = \int_{\bar{\gamma}(x+y, \zeta)}^{x+y} \frac{\partial \tilde{\Gamma}(\eta; z)}{\partial z} d\eta.$$

Therefore,

$$|\partial_z \mathcal{K}(\zeta; x, y, z)| \leq \sup_{\eta, z} \left| \partial_z \tilde{\Gamma}(\eta; z) \right| \int_0^1 |\partial_\zeta \bar{\gamma}(x+y, \zeta \beta)| d\beta |\zeta| \leq \left(\sup_{x, y, z, \eta} \left| \partial_z \tilde{\Gamma}(\eta; z) \right| |\partial_\zeta \bar{\gamma}(x+y, \eta)| \right) |\zeta|,$$

in light of Lemma 2.3. We can similarly prove that $\partial_z^2 H_1(z; x, y)$ exists and is bounded. Using (21) and that $\tilde{\Gamma}(\zeta; z) = \Gamma(\zeta - z; z)$, we get

$$\mathbb{E}H_1(X_s(\varepsilon, \emptyset, x); x, y) = H_{1,0}(x, y) + s\mathcal{R}_s^3(x, y), \quad (63)$$

where $H_{1,0}(x; y) := H_1(x; x+y) = \mathcal{D}_{1,0}(x; y) + \hat{H}_{1,0}(x; y)$ with

$$\begin{aligned} \mathcal{D}_{1,0}(x; y) &:= \Gamma(y; x)b(x+y) - (\partial_\zeta \Gamma)(y; x)v(x+y) - \Gamma(y; x)v'(x+y) \\ \hat{H}_{1,0}(x; y) &:= \int (\mathbb{P}[x + \gamma(x, J) \geq \bar{\gamma}(x+y, \zeta)] - \mathbb{P}[x + \gamma(x, J) \geq x+y]) \bar{h}_\varepsilon(\zeta)d\zeta, \\ \mathcal{R}_s^3(x; y) &:= \int_0^1 \mathbb{E}L_\varepsilon H_1(X_{\alpha s}(\varepsilon, \emptyset, x); x+y)d\alpha = O(1), \quad \text{as } s \rightarrow 0. \end{aligned}$$

In order to handle $\mathbb{P}(X_t(x) \geq x+y | N_t^\varepsilon = 2)$, we again condition on the times of the jumps, which are necessarily distributed as the order statistics of two independent uniform $[0, t]$ random variables. Concretely,

$$\mathbb{P}(X_t(x) \geq x+y | N_t^\varepsilon = 2) = \frac{2}{t^2} \int_0^t \int_{s_1}^t \mathbb{P}(X_t(\varepsilon, \{s_1, s_2\}, x) \geq x+y) ds_2 ds_1. \quad (64)$$

Next, we determine the leading term of $\mathbb{P}(X_t(\varepsilon, \{s_1, s_2\}, x) \geq x+y)$. By conditioning on $\mathcal{F}_{s_2}^-$,

$$\mathbb{P}(X_t(\varepsilon, \{s_1, s_2\}, x) \geq x+y) = \mathbb{E}(G_{t-s_2}(X_{s_2}(\varepsilon, \{s_1\}, x))),$$

where, by Lemma B.1,

$$G_t(z) = \mathbb{P}[X_t(\varepsilon, \emptyset, z + \gamma(z, J)) \geq x+y] = H_0(z; x+y) + tH_1(z; x+y) + t^2 \check{\mathcal{R}}_t(z; x+y). \quad (65)$$

Then, for $\varepsilon > 0$ and t small enough,

$$\begin{aligned} \mathbb{P}(X_t(\varepsilon, \{s_1, s_2\}, x) \geq x + y) &= \mathbb{E}(H_0(X_{s_2}(\varepsilon, \{s_1\}, x); x + y)) \\ &\quad + (t - s_2)\mathbb{E}\mathcal{R}_{t-s_2}^4(X_{s_2}(\varepsilon, \{s_1\}, x); x, y), \\ \text{with } \mathcal{R}_t^4(z; x, y) &:= H_1(z; x + y) + t\check{\mathcal{R}}_t(z; x + y). \end{aligned} \quad (66)$$

Again, conditioning on $\mathcal{F}_{s_1}^-$

$$\mathbb{E}(H_0(X_{s_2}(\varepsilon, \{s_1\}, x); x + y)) = \mathbb{E}\left(\widehat{G}_{s_2-s_1}(X_{s_1}(\varepsilon, \emptyset, x); x + y)\right),$$

where

$$\widehat{G}_t(z; x + y) := \mathbb{E}H_0(X_t(\varepsilon, \emptyset, z + \gamma(z, J)); x + y).$$

Since $z \rightarrow H_0(z; x + y) = \mathbb{P}(z + \gamma(z, J) \geq x + y)$ is C_b^∞ by Lemma 2.3-(3), one can apply (21) to deduce

$$\begin{aligned} \widehat{G}_t(z; x + y) &= \int \widetilde{\Gamma}(\zeta; z) \mathbb{E}H_0(X_t(\varepsilon, \emptyset, \zeta); x + y) d\zeta = \int \widetilde{\Gamma}(\zeta; z) H_0(\zeta; x + y) d\zeta + t\mathcal{R}_t^6(z; x, y) \\ &=: H_2(z; x + y) + t\mathcal{R}_t^6(z; x, y), \end{aligned}$$

where, denoting two independent copies of J by J_1, J_2 ,

$$\begin{aligned} H_2(z; x + y) &:= \mathbb{P}(z + \gamma(z, J_1) + \gamma(z + \gamma(z, J_1), J_2) \geq x + y), \\ \mathcal{R}_t^6(z; x, y) &:= \int \widetilde{\Gamma}(\zeta; z) \int_0^1 \mathbb{E}L_\varepsilon H_0(X_{\alpha t}(\varepsilon, \emptyset, \zeta); x + y) d\alpha d\zeta. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{P}(X_t(\varepsilon, \{s_1, s_2\}, x) \geq x + y) &= \mathbb{E}(H_2(X_{s_1}(\varepsilon, \emptyset, x); x + y)) + (s_2 - s_1)\mathbb{E}\mathcal{R}_{s_2-s_1}^6(X_{s_1}(\varepsilon, \emptyset, x); x, y) \\ &\quad + (t - s_2)\mathbb{E}\mathcal{R}_{t-s_2}^4(X_{s_2}(\varepsilon, \{s_1\}, x); x, y). \end{aligned}$$

Applying again (21) to the first term on the right-hand side of the previous equation,

$$\begin{aligned} \mathbb{P}(X_t(\varepsilon, \{s_1, s_2\}, x) \geq x + y) &= H_{2,0}(x; y) + s_1\mathcal{R}_{s_1}^5(x; y) \\ &\quad + (s_2 - s_1)\mathbb{E}\mathcal{R}_{s_2-s_1}^6(X_{s_1}(\varepsilon, \emptyset, x); x, y) \\ &\quad + (t - s_2)\mathbb{E}\mathcal{R}_{t-s_2}^4(X_{s_2}(\varepsilon, \{s_1\}, x); x, y), \end{aligned} \quad (67)$$

where

$$\begin{aligned} H_{2,0}(x; y) &:= H_2(x; x + y) = \mathbb{P}(\gamma(x, J_1) + \gamma(x + \gamma(x, J_1), J_2) \geq y) \\ \mathcal{R}_{s_1}^5(x; y) &:= \int_0^1 \mathbb{E}L_\varepsilon H_2(X_{\alpha s_1}(\varepsilon, \emptyset, x); x + y) d\alpha \end{aligned}$$

Therefore, we conclude that

$$\mathbb{P}(X_t(x) \geq x + y | N_t^\varepsilon = 2) = H_{2,0}(x; y) + O(t). \quad (68)$$

In light of (53), (56)-(58), (63), and (68), we have the following second order decomposition of the tail distribution $\mathbb{P}(X_t(x) \geq x + y)$:

$$\begin{aligned} \mathbb{P}(X_t(x) \geq x + y) &= e^{-\lambda_\varepsilon t} \lambda_\varepsilon t H_{0,0}(x; y) + e^{-\lambda_\varepsilon t} \frac{\lambda_\varepsilon t^2}{2} (H_{0,1}(x; y) + H_{1,0}(x; y)) \\ &\quad + e^{-\lambda_\varepsilon t} \frac{(\lambda_\varepsilon t)^2}{2} H_{2,0}(x; y) + O(t^3) \\ &= \lambda_\varepsilon t H_{0,0}(x; y) + \frac{t^2}{2} \left\{ \lambda_\varepsilon [H_{0,1}(x; y) + H_{1,0}(x; y)] \right. \\ &\quad \left. + \lambda_\varepsilon^2 [H_{2,0}(x; y) - 2H_{0,0}(x; y)] \right\} + O(t^3). \end{aligned}$$

The expressions in (29) follows from the fact that, for ε small enough,

$$\lambda_\varepsilon \mathbb{P}[\gamma(x, J) \geq y] = \int_y^\infty \lambda_\varepsilon \Gamma_\varepsilon(\zeta; x) d\zeta = \int_{\{\zeta: \gamma(x, \zeta) \geq y\}} h(\zeta) d\zeta = \int_y^\infty g(x; \zeta) d\zeta. \quad (69)$$

Thus, for fixed $x \in \mathbb{R}$ and $y > 0$ and ε small enough,

$$\lambda_\varepsilon \Gamma_\varepsilon(\zeta; x) = g(x; \zeta). \quad (70)$$

Using the two previous relationships, it is not hard to see that

$$\begin{aligned} \lambda_\varepsilon H_{0,0}(x; y) &= \int_y^\infty g(x; \zeta) d\zeta; \quad \lambda_\varepsilon [\hat{H}_{0,1}(x; y) + \hat{H}_{1,0}(x; y)] = \mathcal{J}_1(x; y), \\ \lambda_\varepsilon [H_{0,1}(x; y) + H_{1,0}(x; y)] &= \mathcal{D}(x; y) + \mathcal{J}_1(x; y), \quad \lambda_\varepsilon^2 [H_{2,0}(x; y) - 2H_{0,0}(x; y)] = \mathcal{J}_2(x; y). \end{aligned}$$

This concludes the result of Theorem 4.1.

C Proof of the expansion for the transition densities

The following result will allow us to control the higher order terms (see Appendix D for a proof):

Lemma C.1 *Let*

$$\bar{\mathcal{R}}_t(x, y) := e^{-\lambda_\varepsilon t} \sum_{n=3}^\infty \mathbb{P}(X_t(x) \geq x + y | N_t^\varepsilon = n) \frac{(\lambda_\varepsilon t)^n}{n!}. \quad (71)$$

Then, under the conditions of Theorem 5.8, there exists $\varepsilon > 0$ small enough as well as $t_0 := t_0(\varepsilon) > 0$ and $B = B(\varepsilon) < \infty$ such that, for any $0 < t < t_0$,

$$|\partial_y \bar{\mathcal{R}}_t(x, y)| \leq B t^3.$$

Proof of Theorem 5.8. Let us consider the terms corresponding to one and two “large” jumps in (26). From (53), (56), (57), and (63), it follows that

$$\begin{aligned} \mathbb{P}(X_t(x) \geq x + y | N_t^\varepsilon = 1) &= H_{0,0}(x; y) + \frac{t}{2} [H_{0,1}(x; y) + H_{1,0}(x; y)] \\ &+ \frac{1}{t} \int_0^t \{s^2 \mathcal{R}_s^2(x; y) + (t-s)s \mathcal{R}_s^3(x; y) + (t-s)^2 \mathbb{E} \mathcal{R}_{t-s}^1(X_s(\varepsilon, \emptyset, x); x, y)\} ds. \end{aligned} \quad (72)$$

Similarly, from (64), (66), and (67), we have

$$\begin{aligned} \mathbb{P}(X_t(x) \geq x + y | N_t^\varepsilon = 2) &= H_{2,0}(x; y) \\ &+ \frac{2}{t^2} \int_0^t \int_{s_1}^t \left\{ s_1 \mathcal{R}_{s_1}^5(x; y) + (s_2 - s_1) \mathbb{E} \mathcal{R}_{s_2-s_1}^6(X_{s_1}(\varepsilon, \emptyset, x); x, y) \right. \\ &\quad \left. + (t - s_2) \mathbb{E} \mathcal{R}_{t-s_2}^4(X_{s_2}(\varepsilon, \{s_1\}, x); x, y) \right\} ds_2 ds_1. \end{aligned} \quad (73)$$

Equations (72-73) show that in order for the derivatives

$$\hat{a}_1(x; y) := \frac{\partial}{\partial y} \mathbb{P}(X_t(x) \geq x + y | N_t^\varepsilon = 1), \quad \hat{a}_2(x; y) := \frac{\partial}{\partial y} \mathbb{P}(X_t(x) \geq x + y | N_t^\varepsilon = 2)$$

to exist, it suffices that the partial derivatives of the functions $H_{i,j}(x; y)$ with respect to y exist and also that the partial derivatives of the two types of functions, $\mathcal{R}_t^i(x; y)$ with $i = 2, 3, 5$ and $\mathcal{R}_t^j(w; x, y)$ with $j = 1, 4, 6$, with

respect to y exist and are uniformly bounded on $w \in \mathbb{R}$ and on a neighborhood of y . Furthermore, under the later boundedness property, we will then obtain that

$$\hat{a}_1(x; y) = \frac{\partial H_{0,0}(x; y)}{\partial y} + \frac{t}{2} \left[\frac{\partial H_{0,1}(x; y)}{\partial y} + \frac{\partial H_{1,0}(x; y)}{\partial y} \right] + O(t^2), \quad (t \rightarrow 0), \quad (74)$$

$$\hat{a}_2(x; y) = \frac{\partial H_{2,0}(x; y)}{\partial y} + O(t), \quad (t \rightarrow 0). \quad (75)$$

Note that (74-75) suffices to conclude (34) in light of (26), Theorem 5.7, and Lemma C.1.

(1) Differentiability of $H_{i,j}(x; y)$:

The differentiability of the functions $H_{i,j}(x; y)$ essentially follows from Lemma 2.3. Indeed, Lemma 2.3-(2) implies that $\partial_y H_{0,0}(x; y) = -\Gamma(y; x)$ as well as

$$\begin{aligned} \partial_y H_{0,1}(x; y) &:= \frac{\sigma^2(x)}{2} \left(-\frac{\partial^2 \Gamma(y; x)}{\partial x^2} + 2 \frac{\partial^2 \Gamma(y; x)}{\partial y \partial x} - \frac{\partial^2 \Gamma(y; x)}{\partial y^2} \right) + b(x) \left(-\frac{\partial \Gamma(y; x)}{\partial x} + \frac{\partial \Gamma(y; x)}{\partial y} \right), \\ &+ \int (\Gamma(y; x) - \Gamma(y - \gamma(x, \zeta); x + \gamma(x, \zeta))) \bar{h}_\varepsilon(\zeta) d\zeta. \end{aligned}$$

Similarly, we have that

$$\partial_y H_{1,0}(x; y) := \partial_y \mathcal{D}_{1,0}(x; y) + \int (\Gamma(y; x) - \Gamma(\bar{\gamma}(x + y, \zeta) - x; x) \partial_y \bar{\gamma}(x + y, \zeta)) \bar{h}_\varepsilon(\zeta) d\zeta$$

To compute $\partial_y H_{2,0}(x; y)$, note that

$$\begin{aligned} \frac{\partial}{\partial y} H_{2,0}(x; y) &= \frac{\partial}{\partial y} \int \mathbb{P}(\gamma(x + \gamma(x, \zeta_1), J_2) \geq y - \gamma(x, \zeta_1)) h_\varepsilon(\zeta_1) d\zeta_1 \\ &= \int \frac{\partial}{\partial y} \int_{y - \gamma(x, \zeta_1)}^\infty \Gamma(\zeta_2; x + \gamma(x, \zeta_1)) d\zeta_2 h_\varepsilon(\zeta_1) d\zeta_1 \\ &= - \int \Gamma(y - \gamma(x, \zeta_1); x + \gamma(x, \zeta_1)) h_\varepsilon(\zeta_1) d\zeta_1, \end{aligned}$$

where the second equality above again follows from Lemma 2.3-(2). Finally, the representations in (35) can be deduced for ε small enough from the relationships (69)-(70).

(2) Boundedness of $\partial_y \mathcal{R}^i(w; x, y)$:

Analyzing the remainder terms $\mathcal{R}^2(x; y)$, $\mathcal{R}_t^3(x; y)$, $\mathcal{R}_t^5(x; y)$, and $\mathcal{R}_t^6(w; x, y)$, it transpires that it suffices to show that $\partial_y L_\varepsilon^2 H_0(w; x + y)$, $\partial_y L_\varepsilon H_0(w; x + y)$, $\partial_y L_\varepsilon H_1(w; x + y)$, and $\partial_y L_\varepsilon H_2(w; x + y)$ exist and are uniformly bounded in w and y . From the definition of L_ε in (20), one can see that, for any function $H(w; y) : \mathbb{R}^2 \rightarrow \mathbb{R}$ in $C_b^\infty(\mathbb{R}^2)$, $\partial_y (L_\varepsilon H(w; y))$ exists and

$$\partial_y (L_\varepsilon H(w; y)) = L_\varepsilon (\partial_y H)(w; y), \quad \sup_{w, y} |\partial_y L_\varepsilon H(w; y)| < \infty.$$

As in the proof of Lemma 2.3-(4) and the relationship (62), one can verify that $H_0(w; x + y)$, $H_1(w; x + y)$, $H_2(w; x + y)$ are C_b^∞ functions.

In order to show that $\partial_y \mathcal{R}_t^1(w; x, y)$ and $\partial_y \mathcal{R}_t^4(w; x, y)$ exist and are bounded, it suffices that the remainder term $\check{\mathcal{R}}_t(z; \vartheta)$ of (39) is differentiable with respect to ϑ and $\partial_\vartheta \check{\mathcal{R}}_t(z; \vartheta)$ is bounded. The remainder term is defined as in (47), which in turn is defined as the limit as $\delta \rightarrow 0$ of each of the four terms in (48). We will show that the limit of the second term is indeed differentiable with respect to ϑ and its derivative is bounded. The other three terms can be dealt with similarly. As shown in (52), the limit of second term in (48) turns out to be $\bar{I}_t^{(2,1)}(\vartheta; z, \varepsilon) + \bar{I}_t^{(2,2)}(\vartheta; z, \varepsilon)$ and, thus, it suffices to show that the functions $\bar{I}_t^{(2,1)}(w; z, \varepsilon)$ and $\bar{I}_t^{(2,2)}(w; z, \varepsilon)$ defined in (49) are differentiable with respect to w and that their respective derivatives are bounded. The latter two conditions will follow from Lemma B.2 together with the same arguments leading to (50)-(51). ■

D Proofs of other lemmas and additional needed results

Proof of Lemma 2.3. In light of **(C2-b)** and the continuity of $(z, \zeta) \rightarrow \partial_\zeta \gamma(z, \zeta)$, we have that either $\partial_\zeta \gamma(z, \zeta) \leq -\delta$ or $\partial_\zeta \gamma(z, \zeta) \geq \delta$, for all (z, ζ) . Without loss of generality, we assume throughout the proof the latter case (the other case can be treated in the same way). Since $\partial_\zeta \gamma(z, \zeta) \geq \delta$, the mappings $\zeta \rightarrow \tilde{\gamma}(z, \zeta)$ and $\zeta \rightarrow \gamma(z, \zeta)$ are both strictly increasing for all $z \in \mathbb{R}$ and their inverse exists. By the same argument, we can assume without loss of generality that $1 + \partial_x \gamma(x, \zeta) \geq \delta$ for all x, ζ and, thus, the mapping $z \rightarrow z + \gamma(z, \zeta) =: u$ is strictly increasing and its inverse $\tilde{\gamma}(u, \zeta)$ exists. We now show the different assertions of the lemma:

(1) Let us just consider $\tilde{\gamma}^{-1}(z, \zeta)$ (the verification for $\gamma^{-1}(z, \zeta)$ goes along the same lines). Since $\tilde{\gamma}(z, \zeta) = \zeta(z, \zeta) + z$ is continuously differentiable and condition **(C2-b)** is satisfied, the Inverse Function Theorem implies $\tilde{\gamma}^{-1}(z, \zeta)$ is differentiable in ζ and

$$\frac{\partial \tilde{\gamma}^{-1}}{\partial \zeta}(z, \zeta) = \left(\frac{\partial \tilde{\gamma}}{\partial \zeta}(z, \tilde{\gamma}^{-1}(z, \zeta)) \right)^{-1} = \left(\frac{\partial \gamma}{\partial \zeta}(z, \tilde{\gamma}^{-1}(z, \zeta)) \right)^{-1}. \quad (76)$$

Moreover, since $\tilde{\gamma}$ is $C^\infty(\mathbb{R} \times \mathbb{R})$, the Implicit Function Theorem implies that $\tilde{\gamma}^{-1}(z, \zeta)$ is $C^\infty(\mathbb{R} \times \mathbb{R})$. By implicitly differentiating the equation $\tilde{\gamma}^{-1}(z, \tilde{\gamma}(z, \zeta)) = \zeta$ with respect to z and then using (76), we get

$$\begin{aligned} \frac{\partial \tilde{\gamma}^{-1}}{\partial z}(z, \zeta) &= -\frac{\partial \tilde{\gamma}}{\partial z}(z, \tilde{\gamma}^{-1}(z, \zeta)) \left(\frac{\partial \tilde{\gamma}}{\partial \zeta}(z, \tilde{\gamma}^{-1}(z, \zeta)) \right)^{-1} \\ &= -\left(\frac{\partial \gamma}{\partial z}(z, \tilde{\gamma}^{-1}(z, \zeta)) + 1 \right) \left(\frac{\partial \gamma}{\partial \zeta}(z, \tilde{\gamma}^{-1}(z, \zeta)) \right)^{-1}. \end{aligned} \quad (77)$$

By formally differentiating the expressions (76) and (77), it follows that $\tilde{\gamma}(z, \zeta)$ is $C_b^{\geq 1}$. Indeed, by induction, one can verify that for any $k, \ell \geq 0$ with $k + \ell \geq 1$,

$$\frac{\partial^{k+\ell} \tilde{\gamma}^{-1}}{\partial z^k \partial \zeta^\ell}(z, \tilde{\gamma}^{-1}(z, \zeta)) = \left(\frac{\partial \tilde{\gamma}}{\partial \zeta}(z, \zeta) \right)^{-k-\ell} \Phi \left(\frac{\partial^{k'+\ell'} \tilde{\gamma}}{\partial z^{k'} \partial \zeta^{\ell'}}(z, \zeta) : 1 \leq k' + \ell' \leq k + \ell \right),$$

where $\Phi \left(\frac{\partial^{k'+\ell'} \tilde{\gamma}}{\partial z^{k'} \partial \zeta^{\ell'}}(z, \zeta) : k' + \ell' \leq k + \ell, k' + \ell' \geq 1 \right)$ represents a polynomial function of partial derivatives of $\tilde{\gamma}$ of order greater than 0 but no greater than $k + \ell$. Since $\tilde{\gamma}(z, \zeta) \in C_b^{\geq 1}$ and $|\partial_\zeta \tilde{\gamma}(z, \zeta)| = |\partial_\zeta \gamma(z, \zeta)| \geq \delta$, we conclude that $\tilde{\gamma}^{-1}(z, \zeta) \in C_b^{\geq 1}$ as well.

(2) Since $\zeta \rightarrow \tilde{\gamma}(z, \zeta)$ is assumed to be strictly increasing,

$$\mathbb{P}(\tilde{\gamma}(z, J^\varepsilon) \geq \zeta) = \mathbb{P}(J^\varepsilon \geq \tilde{\gamma}^{-1}(z, \zeta)) = \int_{\tilde{\gamma}^{-1}(z, \zeta)}^\infty \check{h}_\varepsilon(v) dv = \bar{F}_\varepsilon(\tilde{\gamma}^{-1}(z, \zeta)), \quad (78)$$

where $\bar{F}_\varepsilon(u) = \int_u^\infty \check{h}_\varepsilon(v) dv$. Clearly, the function $(z, \zeta) \rightarrow \mathbb{P}(\tilde{\gamma}(z, J^\varepsilon) \geq \zeta)$ is $C_b^\infty(\mathbb{R} \times \mathbb{R})$ (being the composition of two C_b^∞ functions). In particular, by (76),

$$\frac{\partial}{\partial \zeta} \mathbb{P}(\tilde{\gamma}(z, J^\varepsilon) \geq \zeta) = \bar{F}'_\varepsilon(\tilde{\gamma}^{-1}(z, \zeta)) \frac{\partial \tilde{\gamma}^{-1}}{\partial \zeta}(z, \zeta) = -\check{h}_\varepsilon(\tilde{\gamma}^{-1}(z, \zeta)) \left(\frac{\partial \tilde{\gamma}}{\partial \zeta}(z, \tilde{\gamma}^{-1}(z, \zeta)) \right)^{-1},$$

which in turn leads to the existence of $\tilde{\Gamma}(\zeta; z)$. This density admits the representation:

$$\tilde{\Gamma}(\zeta; z) = \check{h}_\varepsilon(\tilde{\gamma}^{-1}(z, \zeta)) \left(\frac{\partial \tilde{\gamma}}{\partial \zeta}(z, \tilde{\gamma}^{-1}(z, \zeta)) \right)^{-1},$$

which is of the form (15) since $\partial_\zeta \tilde{\gamma}(z, \zeta) = \partial_\zeta \gamma(z, \zeta)$ was assumed to be positive. Being $\check{h}_\varepsilon \in C_b^\infty$ and $\partial \tilde{\gamma} / \partial \zeta \in C_b^\infty$ and greater than δ , it is clear that $\tilde{\Gamma} \in C_b^\infty$. The density for $\gamma(z, J^\varepsilon)$ can be treated in the same way since $\zeta \rightarrow \gamma(z, J^\varepsilon)$ is also strictly increasing.

(3) Differentiating both sides of (78) with respect to z and using (77),

$$\begin{aligned}\frac{\partial}{\partial z} \mathbb{P}(\tilde{\gamma}(z, J^\varepsilon) \geq \zeta) &= -\check{h}_\varepsilon(\tilde{\gamma}^{-1}(z, \zeta)) \frac{\partial \tilde{\gamma}^{-1}}{\partial z}(z, \zeta) \\ &= \check{h}_\varepsilon(\tilde{\gamma}^{-1}(z, \zeta)) \frac{\partial \tilde{\gamma}}{\partial z}(z, \tilde{\gamma}^{-1}(z, \zeta)) \left(\frac{\partial \tilde{\gamma}}{\partial \zeta}(z, \tilde{\gamma}^{-1}(z, \zeta)) \right)^{-1}.\end{aligned}$$

Being $\check{h}_\varepsilon \in C_b^\infty$ and $\partial \tilde{\gamma} / \partial \zeta \in C_b^\infty$ and greater than δ , it is clear that $\mathbb{P}(\tilde{\gamma}(z, J^\varepsilon) \geq \zeta) \in C_b^\infty$. In general, we will have that

$$\frac{\partial}{\partial z} \mathbb{P}(\tilde{\gamma}(z, J^\varepsilon) \geq \zeta) = \check{h}_\varepsilon(\tilde{\gamma}^{-1}(z, \zeta)) \left| \frac{\partial \tilde{\gamma}}{\partial \zeta}(z, \tilde{\gamma}^{-1}(z, \zeta)) \right|^{-1} \left(1 + \frac{\partial \gamma}{\partial z}(z, \tilde{\gamma}^{-1}(z, \zeta)) \right), \quad (79)$$

One can similarly show that $(z, \zeta) \rightarrow \mathbb{P}(\gamma(z, J^\varepsilon) \geq \zeta)$ is C_b^∞ .

(4) The proof goes along the same lines as that of (1) above. Note that in this case,

$$\frac{\partial \tilde{\gamma}}{\partial u}(u, \zeta) = \frac{1}{1 + (\partial_x \gamma)(\tilde{\gamma}(u, \zeta), \zeta)}, \quad \frac{\partial \tilde{\gamma}}{\partial \zeta}(u, \zeta) = -\frac{(\partial_\zeta \gamma)(\tilde{\gamma}(u, \zeta), \zeta)}{1 + (\partial_x \gamma)(\tilde{\gamma}(u, \zeta), \zeta)}. \quad (80)$$

■ **Proof of Lemma 3.3.** Recall that the decomposition (20) where for simplicity hereafter we shall drop the subindex in the operators \mathcal{I}_ε and \mathcal{D}_ε . In order to ease the notation, we write $Y_t := X_t(\varepsilon, \emptyset, x)$ below. Applying Itô's formula, we have

$$f(Y_t) = f(x) + \int_0^t L_\varepsilon f(Y_u) du + \int_0^t f'(Y_u) \sigma(Y_u) dW_u + \int_0^t \int \mathcal{B}(Y_{u-}, \zeta) \bar{M}'(du, d\zeta),$$

where \bar{M}' is the compensated Poisson measure of $Z'(\varepsilon)$ and $\mathcal{B}(y, \zeta) := f(y + \gamma(y, \zeta)) - f(y)$. Since $f \in C_b^1$ and σ is bounded, the second integral above is a martingale. The last integral is also a martingale. Indeed, $\int_0^t \int_{|\zeta| \geq 1} \mathcal{B}(Y_{u-}, \zeta) \bar{M}'(du, d\zeta)$ is a martingale because f is bounded, while $\int_0^t \int_{|\zeta| < 1} \mathcal{B}(Y_{u-}, \zeta) \bar{M}'(du, d\zeta)$ is a martingale because

$$\int_0^t \int_{|\zeta| < 1} \mathcal{B}(Y_{u-}, \zeta)^2 \bar{h}_\varepsilon(\zeta) d\zeta du \leq \int_0^t \int_{|\zeta| < 1} \int_0^1 |f'(Y_u + \gamma(Y_u, \zeta)\beta)|^2 d\beta |\gamma(Y_u, \zeta)|^2 \bar{h}_\varepsilon(\zeta) d\zeta du < \infty,$$

due to condition **(C2-a)**. Then, we conclude that

$$\mathbb{E}f(Y_t) = f(x) + \mathbb{E} \int_0^t L_\varepsilon f(Y_u) du. \quad (81)$$

Next, we show that $L_\varepsilon f(y)$ is bounded. Obviously, $\mathcal{D}f(y)$ is bounded whenever $f \in C_b^2$ and **(C3)** hold. Also,

$$|\mathcal{I}f(y)| = \left| \int (f(y + \gamma(y, \zeta)) - f(y)) \bar{h}_\varepsilon(\zeta) d\zeta \right| \leq \int \int_0^1 |f'(y + \gamma(y, \zeta)\beta)| d\beta |\gamma(y, \zeta)| \bar{h}_\varepsilon(\zeta) d\zeta,$$

which is clearly bounded in y due to condition **(C2-a)**. We can then interchange expectation and integration in (81). (21) follows from the change of variables $u = \alpha t$.

We now proceed to show (23). To this end, we check that $L_\varepsilon f$ belongs to C_b^2 if $f \in C_b^4$. Clearly, $\mathcal{D}f \in C_b^2$ in light of **(C3)**. To show that $\mathcal{I}f \in C_b^2$, note that

$$\begin{aligned}|\partial_y [f(y + \gamma(y, \zeta)) - f(y)]| &= |f'(y + \gamma(y, \zeta))(1 + \partial_y \gamma(y, \zeta)) - f'(y)| \\ &\leq \|f''\|_\infty |\gamma(y, \zeta)| + \|f'\|_\infty |\partial_y \gamma(y, \zeta)|,\end{aligned}$$

which can be bounded by $C_\varepsilon(\|f''\|_\infty + \|f'\|_\infty)|\zeta|$ for all $|\zeta| \leq \varepsilon$ in light of **(C2-a)**. Given that $\int |\zeta| \bar{h}_\varepsilon(\zeta) d\zeta < \infty$, we can interchange differentiation and integration to get

$$\begin{aligned}(\mathcal{I}f)'(y) &= \int (f'(y + \gamma(y, \zeta))(1 + \partial_y \gamma(y, \zeta)) - f'(y)) \bar{h}_\varepsilon(\zeta) d\zeta \\ &= \int \int_0^1 f''(y + \gamma(y, \zeta)\beta) d\beta \gamma(y, \zeta) \bar{h}_\varepsilon(\zeta) d\zeta + \int f'(y + \gamma(y, \zeta)) \partial_y \gamma(y, \zeta) \bar{h}_\varepsilon(\zeta) d\zeta.\end{aligned}$$

In particular, $\sup_y |(\mathcal{I}f)'(y)| \leq C_\varepsilon(\|f''\|_\infty + \|f'\|_\infty) \int |\zeta| \bar{h}_\varepsilon(\zeta) d\zeta =: K$ and

$$\sup_y |(L_\varepsilon f)'(y)| \leq \|b'\|_\infty \|f'\|_\infty + \|b\|_\infty \|f''\|_\infty + \|v'\|_\infty \|f''\|_\infty + \|v\|_\infty \|f^{(3)}\|_\infty + K.$$

We can similarly show that $\sup_y |(L_\varepsilon f)''(y)| \leq K$ when $f \in C_b^4$, for a constant $K < \infty$ depending only on $\|f^{(i)}\|_\infty$, $\|v^{(i)}\|_\infty$, $\|b^{(i)}\|_\infty$, for $i = 1, \dots, 4$, and $\int |\zeta| \bar{h}_\varepsilon(\zeta) d\zeta$. We also have

$$\begin{aligned} L_\varepsilon^2 f(y) &= b(y)(L_\varepsilon f)'(y) + \frac{\sigma^2(y)}{2} (L_\varepsilon f)''(y) + \int [L_\varepsilon f(y + \gamma(y, \zeta)) - L_\varepsilon f(y)] \bar{h}_\varepsilon(\zeta) d\zeta \\ &= b(y)(L_\varepsilon f)'(y) + \frac{\sigma^2(y)}{2} (L_\varepsilon f)''(y) + \int \int_0^1 (L_\varepsilon f)'(y + \gamma(y, \zeta)\beta) d\beta \gamma(y, \zeta) \bar{h}_\varepsilon(\zeta) d\zeta. \end{aligned}$$

Therefore,

$$\|L_\varepsilon^2 f\|_\infty \leq \|b\|_\infty \|(L_\varepsilon f)'\|_\infty + \|v\|_\infty \|(L_\varepsilon f)''\|_\infty + C_\varepsilon \|(L_\varepsilon f)'\|_\infty \int |\zeta| \bar{h}_\varepsilon(\zeta) d\zeta < \infty. \quad (82)$$

Since $L_\varepsilon f \in C_b^2$, by interchanging expectation and integration in (81) and applying again Dynkin's formula, we get

$$\begin{aligned} \mathbb{E}f(Y_t) &= f(x) + \int_0^t \mathbb{E}L_\varepsilon f(Y_u) du = f(x) + \int_0^t \left(L_\varepsilon f(x) + \int_0^u \mathbb{E}L_\varepsilon^2 f(Y_v) dv \right) du \\ &= f(x) + tL_\varepsilon f(x) + \int_0^t \int_0^u \mathbb{E}L_\varepsilon^2 f(Y_v) dv du. \end{aligned}$$

Applying Fubini in the last integral and changing variable to $\alpha = v/t$, we finally have

$$\mathbb{E}f(Y_t) = f(x) + tL_\varepsilon f(x) + t^2 \int_0^1 (1 - \alpha) \mathbb{E}L_\varepsilon^2 f(Y_{\alpha t}) d\alpha.$$

■

The following result is needed in order to prove Lemma B.2.

Lemma D.1 *Assume the Conditions (C1-C4) of Section 2 are enforced and let $\Phi_t : x \rightarrow X_t(\varepsilon, \emptyset, x)$ be the diffeomorphism associated with the solution of the SDE (16). Then, for any $k \geq 1$ and compact $K \subset \mathbb{R}$,*

$$\limsup_{t \rightarrow 0} \sup_{\eta \in K} \mathbb{E} \left(\left| \frac{d^i \Phi_t^{-1}}{d\eta^i}(\eta) \right|^k \right) < \infty, \quad i = 1, 2. \quad (83)$$

Proof. To simplify the notation, we write $\check{X}(x) = \{\check{X}_t(x)\}_{t \geq 0}$ for $\{X_t(\varepsilon, \emptyset, x)\}_{t \geq 0}$ and fix $Y_s(x) := \check{X}_{(t-s)-}(x)$ for $0 \leq s < t$ and $Y_t(x) := \check{X}_0(x) = x$. We follow a similar approach to that in the proof of Lemma 3.1 in [18] based on time-reversibility (see Section VI.4 in [30] for further information). Recall that the time-reversal process of a càdlàg process $V = \{V_s\}_{0 \leq s \leq t}$ is given by the càdlàg process

$$\bar{V}_s^t = 0\mathbf{1}_{s=0} + (V_{(t-s)-} - V_{t-})\mathbf{1}_{0 < s < t} + (V_0 - V_{t-})\mathbf{1}_{s=t}. \quad (84)$$

Our main tool is Theorem VI.4.22 in [30]. The following notation and definitions are useful for verifying the assumptions in the theorem.

Throughout, $\Phi_{s,t}(\cdot; \omega) : \mathbb{R} \rightarrow \mathbb{R}$ denotes the diffeomorphisms defined by $\Phi_{s,t}(x; \omega) := X_{s,t}^\varepsilon(x; \omega)$ where $X_{s,t}^\varepsilon(x; \omega)$ is the unique solution of the SDE

$$X_{s,t}^\varepsilon(x) = x + \int_s^t \sigma(X_{s,u}^\varepsilon(x)) dW_u + \int_s^t b(X_{s,u}^\varepsilon(x)) du + \sum_{s < u \leq t} \gamma(X_{s,u-}^\varepsilon(x), \Delta Z'_u). \quad (85)$$

The a.s. existence of this diffeomorphisms is guaranteed from (11) as stated in Remark 2.2. As usual, $\mathcal{F}_s = \mathcal{F}_s^0 \vee \mathcal{N}$ and $\mathbb{F} = (\mathcal{F}_s)_{0 \leq s \leq t}$, where $\mathcal{F}_s^0 = \sigma\{W_u, Z'_u; u \leq s\}$ ($0 \leq s \leq t$) and \mathcal{N} are the \mathbb{P} -null sets of \mathcal{F}_t^0 . We also define

the backward filtration $\tilde{\mathbb{H}} = (\mathcal{H}^s)_{0 \leq s \leq t}$ by $\mathcal{H}^s = \bigcap_{s < u \leq t} \bar{\mathcal{F}}_u \vee \sigma\{\check{X}_t\}$, where $(\bar{\mathcal{F}}_s)_{0 \leq s \leq t}$ is defined analogously to $(\mathcal{F}_s)_{0 \leq s \leq t}$ with W and Z' replaced with their reversal processes \bar{W}^t and \bar{Z}'^t .

We are ready to show the assertions of the Lemma. First, note that, by the uniqueness of the solution of (85), $\check{X}_t(x) = \Phi_{s,t}(\check{X}_s(x))$. Thus, $\check{X}_s(x) = \Phi_{s,t}^{-1}(\check{X}_t(x)) \in \mathcal{H}^{t-s}$ and, of course, $\check{X}_s(x) \in \mathcal{F}_s$, so that $\sigma(\check{X}_s(x)) \in \mathcal{F}_s \wedge \mathcal{H}^{t-s}$. Also, by Itô's formula, the quadratic covariation of $W = \{W_s\}_{0 \leq s \leq t}$ with $\sigma(\check{X}) := \{\sigma(\check{X}_s(x))\}_{0 \leq s \leq t}$ is given by

$$\left[\sigma(\check{X}), W\right]_s = \int_0^s \sigma'(\check{X}_u(x))\sigma(\check{X}_u(x))du = \int_0^s \sigma'(Y_{t-u}(x))\sigma(Y_{t-u}(x))du. \quad (86)$$

Finally, recalling that $W = \{W_s\}_{0 \leq s \leq t}$ is an $(\mathbb{F}, \tilde{\mathbb{H}})$ -reversible semimartingale (cf. Theorem VI.4.20 in [30]), the assumptions of Theorem VI.4.22 in [30] are satisfied with $\sigma(\check{X})$ and W in place of H and Y , respectively. By the theorem, we have

$$\overline{\int_0^s \sigma(\check{X}_u(x))dW_u}_s^t + \left[\sigma(\check{X}), W\right]_s^t = \int_0^s \sigma(\check{X}_{t-u}(x))d\bar{W}_u^t,$$

or equivalently, by (86) and the change of variable $v = t - u$,

$$\overline{\int_0^s \sigma(\check{X}_{t-u}(x))dW_u}_s^t - \int_0^s \sigma'(Y_v(x))\sigma(Y_v(x))dv = \int_0^s \sigma(Y_v(x))d\bar{W}_v^t. \quad (87)$$

Omitting for simplicity the dependence of the processes on x , the first term on the left-hand-side of (87) can be written as

$$\begin{aligned} \overline{\check{X}_s - x - \int_0^s b(\check{X}_{u-})du - \sum_{0 < u \leq s} \gamma(\check{X}_{u-}, \Delta Z'_u)}_s^t &= \check{X}_{(t-s)-} - \check{X}_{t-} + \int_{t-s}^t b(\check{X}_u)du + \sum_{t-s \leq u < t} \gamma(\check{X}_{u-}, \Delta Z'_u) \\ &= Y_s - Y_0 + \int_0^s b(Y_v)dv + \sum_{0 < v \leq s} \gamma(\check{X}_{(t-v)-}, \Delta Z'_{t-v}), \end{aligned}$$

where the last equality above is from the change of variable $v = t - u$. Then, (87) implies that

$$\begin{aligned} Y_s(x) &= Y_0(x) - \int_0^s b(Y_v(x))dv + \int_0^s \sigma'(Y_v(x))\sigma(Y_v(x))dv + \int_0^s \sigma(Y_v(x))d\bar{W}_v^t \\ &\quad - \sum_{0 < v \leq s} \gamma(\check{X}_{(t-v)-}(x), \Delta Z'_{t-v}), \quad Y_0(x) = \check{X}_{t-}(x). \end{aligned}$$

Let us write the jump component of Y in a more convenient way. To this end, note that, since $\check{X}_{(t-v)-}(x) + \gamma(\check{X}_{(t-v)-}(x), \Delta Z'_{t-v}) = \check{X}_{t-v}(x)$, one can express $\check{X}_{(t-v)-}(x)$ in terms of the inverse $\bar{\gamma}(u, \zeta)$ of the mapping $z \rightarrow u := z + \gamma(z, \zeta)$ as follows

$$Y_v(x) = \check{X}_{(t-v)-}(x) = \bar{\gamma}(\check{X}_{t-v}(x), \Delta Z'_{t-v}) = \bar{\gamma}(Y_{v-}(x), \Delta Z'_{t-v}).$$

Then,

$$\Delta Y_v(x) = \bar{\gamma}(Y_{v-}(x), \Delta Z'_{t-v}) - Y_{v-}(x) = \bar{\gamma}(Y_{v-}(x), -\Delta \bar{Z}'_v) - Y_{v-}(x) = \gamma_0(Y_{v-}(x), \Delta \bar{Z}'_v),$$

where $\gamma_0(u, \zeta) := \bar{\gamma}(u, -\zeta) - u$ and $\bar{Z}'_v := \bar{Z}'_v^t$ is the time-reversal process of $\{Z'_v\}_{0 \leq v \leq t}$. We conclude that

$$Y_s(x) = \check{X}_{t-}(x) - \int_0^s b(Y_v(x))dv + \int_0^s \sigma'(Y_v(x))\sigma(Y_v(x))dv + \int_0^s \sigma(Y_v(x))d\bar{W}_v^t + \sum_{0 < v \leq s} \gamma_0(Y_{v-}(x), \Delta \bar{Z}'_v).$$

Now, define the diffeomorphism $\Psi_s : \mathbb{R} \rightarrow \mathbb{R}$ as $\Psi_s(\eta) := \check{Y}_s(\eta)$, where $\{\check{Y}_s(\eta)\}_{0 \leq s \leq t}$ is the solution of the SDE

$$\check{Y}_s(\eta) = \eta - \int_0^s b(\check{Y}_v(\eta))dv + \int_0^s \sigma'(\check{Y}_v(\eta))\sigma(\check{Y}_v(\eta))dv + \int_0^s \sigma(\check{Y}_v(\eta))d\bar{W}_v^t + \sum_{0 < v \leq s} \gamma_0(\check{Y}_{v-}(\eta), \Delta \bar{Z}'_v).$$

Since, \mathbb{P} -a.s.,

$$\Psi_t(\Phi_t(x)) = \Psi_t(\check{X}_t(x)) = \Psi_t(\check{X}_{t-}(x)) = Y_t(x) = x, \quad \text{for all } x,$$

it follows that, \mathbb{P} -a.s., $\Psi_t(\eta) = \Phi_t^{-1}(\eta)$ for all $\eta \in \mathbb{R}$. Furthermore, $\{\check{Y}_t(\eta)\}_{t \geq 0}$ solves an SDE of the form (6-2) in [7] with their coefficients satisfying the assumptions of Lemma 10-29 therein. Finally, by Lemma 10-2-c in [7], with $n = 2$ and $q = 1$,

$$\sup_{0 < s \leq t} \sup_{\eta \in K} \mathbb{E} \left[\left| \frac{d^i \Phi_s^{-1}(\eta)}{d\eta^i} \right|^k \right] = \sup_{0 < s \leq t} \sup_{\eta \in K} \mathbb{E} \left[\left| \frac{d^i \Psi_s(\eta)}{d\eta^i} \right|^k \right] = \sup_{0 < s \leq t} \sup_{\eta \in K} \mathbb{E} \left[\left| \frac{d^i \check{Y}_s(\eta)}{d\eta^i} \right|^k \right] < \infty, \quad (i = 1, 2).$$

■ **Proof of Lemma B.2.** For simplicity, we write $\tilde{\Gamma}(\zeta) = \tilde{\Gamma}(\zeta; z)$ and only show the case $k = 1$ (the other cases can similarly be proved). Using the same ideas as in the proof of Proposition I.2 in [19], one can show that

$$\int \tilde{\Gamma}(\zeta) p_t(\eta; \varepsilon, \emptyset, \zeta) d\zeta = \mathbb{E}(H_t(\eta)),$$

where

$$H_t(\eta) := \tilde{\Gamma}(\Phi_t^{-1}(\eta)) \frac{d\Phi_t^{-1}}{d\eta}(\eta)$$

Denoting $\bar{J}_t(\eta) := d\Phi_t^{-1}(\eta)/d\eta$, note that

$$H'_t(\eta) = \tilde{\Gamma}'(\Phi_t^{-1}(\eta)) \bar{J}_t(\eta)^2 + \tilde{\Gamma}(\Phi_t^{-1}(\eta)) \bar{J}'_t(\eta),$$

and, using (83) and that $\tilde{\Gamma} \in C_b^\infty$, it follows that $\sup_{\eta \in K} \mathbb{E} |H'_t(\eta)|^2 < \infty$. In particular,

$$\lim_{h \rightarrow 0} \mathbb{E} \left(\frac{H_t(\eta + h) - H_t(\eta)}{h} \right) = \mathbb{E} \left(\lim_{h \rightarrow 0} \frac{H_t(\eta + h) - H_t(\eta)}{h} \right) = \mathbb{E} H'_t(\eta), \quad (88)$$

since the set of random variables $\{[H_t(\eta + h) - H_t(\eta)]/h : |h| < 1\}$ is uniformly integrable. Indeed,

$$\sup_{|h| \leq 1} \mathbb{E} \left(\frac{H_t(\eta + h) - H_t(\eta)}{h} \right)^2 = \sup_{|h| \leq 1} \mathbb{E} \left(\int_0^1 H'_t(\eta + h\beta) d\beta \right)^2 \leq \sup_{\substack{|h| \leq 1 \\ \beta \in [0,1]}} \mathbb{E} (H'_t(\eta + h\beta))^2 < \infty,$$

again due to (83). Then, (88) can be written as

$$\frac{d}{d\eta} \int \tilde{\Gamma}(\zeta) p_t(\eta; \varepsilon, \emptyset, \zeta) d\zeta = \mathbb{E} \left(\tilde{\Gamma}'(\Phi_t^{-1}(\eta)) (\bar{J}_t(\eta))^2 \right) + \mathbb{E} \left(\tilde{\Gamma}(\Phi_t^{-1}(\eta)) \bar{J}'_t(\eta) \right).$$

It is now clear that (41) will hold true in light of (83). ■

Lemma D.2 Assume the conditions **(C1)**-(**C4**) of Section 2 are satisfied and let \mathcal{D}_ε and \mathcal{I}_ε be the operators defined in (20). Define their respective dual operators $\tilde{\mathcal{D}}_\varepsilon$ and $\tilde{\mathcal{I}}_\varepsilon$ as

$$\begin{aligned} \tilde{\mathcal{D}}_\varepsilon g(y) &:= v(y)g''(y) + (2v'(y) - b(y))g'(y) + (v''(y) - b'(y))g(y), \\ \tilde{\mathcal{I}}_\varepsilon g(y) &:= \int (g(\bar{\gamma}(y, \zeta)) \partial_y \bar{\gamma}(y, \zeta) - g(y)) \bar{h}_\varepsilon(\zeta) d\zeta \\ &= \int (g(\bar{\gamma}(y, \zeta)) - g(y) - g(y)(\partial_y \bar{\gamma})(\bar{\gamma}(y, \zeta), \zeta)) \frac{\bar{h}_\varepsilon(\zeta)}{1 + (\partial_y \bar{\gamma})(\bar{\gamma}(y, \zeta), \zeta)} d\zeta, \end{aligned}$$

where hereafter $\bar{\gamma}(u, \zeta)$ denotes the inverse of the mapping $y \rightarrow u := y + \gamma(y, \zeta)$ for a fixed ζ and whose existence is guaranteed from condition **(C4)**. Then, the following assertions hold:

1. $\tilde{\mathcal{D}}_\varepsilon g$ is well-defined and uniformly bounded for any $g \in C_b^2$ and, furthermore, for any $f \in C_b^2$ with compact support,

$$\int g(y) \mathcal{D}_\varepsilon f(y) dy = \int f(y) \tilde{\mathcal{D}}_\varepsilon g(y) dy. \quad (89)$$

2. $\tilde{\mathcal{I}}_\varepsilon g$ is well-defined and uniformly bounded for any $g \in C_b^1$ and, additionally, if g is integrable, then, for any $f \in C_b^1$,

$$\int g(y) \mathcal{I}_\varepsilon f(y) dy = \int f(y) \tilde{\mathcal{I}}_\varepsilon g(y) dy. \quad (90)$$

3. For any $g \in C_b^1$ and $f \in C_b^1$ with compact support,

$$\int g(y) \mathcal{I}_\varepsilon f(y) dy = \int f'(y) \int \int_{\tilde{\gamma}(y, \zeta)}^y g(\eta) d\eta \bar{h}_\varepsilon(\zeta) d\zeta dy. \quad (91)$$

Proof. The dual relationships essentially follow from a combination of integration by parts and change of variables. Let us show (91). First, note that

$$\begin{aligned} \int g(y) \mathcal{I}_\varepsilon f(y) dy &= \int g(y) \int \int_0^1 f'(y + \gamma(y, \zeta \beta)) (\partial_\zeta \gamma)(y, \zeta \beta) d\beta \zeta \bar{h}_\varepsilon(\zeta) d\zeta dy \\ &= \int \int \int_0^1 g(y) f'(y + \gamma(y, \zeta \beta)) (\partial_\zeta \gamma)(y, \zeta \beta) d\beta \bar{h}_\varepsilon(\zeta) \zeta d\zeta dy. \end{aligned}$$

Changing variable from y to $w := \tilde{\gamma}(y, \zeta \beta) = y + \gamma(y, \zeta \beta)$ and applying Fubini, we get

$$\int g(y) \mathcal{I}_\varepsilon f(y) dy = \int f'(w) \int \int_0^1 g(\tilde{\gamma}(w, \zeta \beta)) \frac{(\partial_\zeta \gamma)(\tilde{\gamma}(w, \beta \zeta), \zeta \beta)}{1 + (\partial_y \gamma)(\tilde{\gamma}(w, \beta \zeta), \zeta \beta)} d\beta \zeta \bar{h}_\varepsilon(\zeta) d\zeta dw.$$

Finally, the second equality in (91) follows from the identity:

$$\partial_\zeta \int_{\tilde{\gamma}(w, \zeta)}^w g(\eta) d\eta = -g(\tilde{\gamma}(w, \zeta)) \partial_\zeta \tilde{\gamma}(w, \zeta) = g(\tilde{\gamma}(w, \zeta)) \frac{(\partial_\zeta \gamma)(\tilde{\gamma}(w, \zeta), \zeta)}{1 + (\partial_y \gamma)(\tilde{\gamma}(w, \zeta), \zeta)}.$$

■

Proof of Lemma C.1. By conditioning on the times of the jumps, which are necessarily distributed as the order statistics of n independent uniform $[0, t]$ random variables, we have

$$\mathbb{P}(X_t(x) \geq x + y | N_t^\varepsilon = n) = \frac{n!}{t^n} \int_\Delta \mathbb{P}(X_t(\varepsilon, \{s_1, \dots, s_n\}, x) \geq x + y) ds_n \dots ds_1,$$

where $\Delta := \{(s_1, \dots, s_n) : 0 < s_1 < s_2 < \dots < s_n < t\}$. Hence, conditioning on $\mathcal{F}_{s_n^-}$,

$$\begin{aligned} \mathbb{P}(X_t(\varepsilon, \{s_1, \dots, s_n\}, x) \geq x + y) &= \mathbb{E} \left[\mathbb{P} \left(X_t(\varepsilon, \{s_1, \dots, s_n\}, x) \geq x + y | \mathcal{F}_{s_n^-} \right) \right] \\ &= \mathbb{E} [G_{t-s_n}(X_{s_n}(\varepsilon, \{s_1, \dots, s_{n-1}\}, x); x, y)], \end{aligned}$$

where $G_t(z; x, y) = \mathbb{P}(X_t(\varepsilon, \emptyset, z + \gamma(z, J)) \geq x + y)$. In terms of the respective densities $p_t(\cdot; \varepsilon, \emptyset, \zeta)$ and $\tilde{\Gamma}(\cdot; z)$ of $X_t(\varepsilon, \emptyset, \zeta)$ and $z + \gamma(z, J)$, we obviously have that

$$G_t(z; x, y) = \int \int_{x+y}^\infty p_t(\eta; \varepsilon, \emptyset, \zeta) d\eta \tilde{\Gamma}(\zeta; z) d\zeta = \int_{x+y}^\infty \int p_t(\eta; \varepsilon, \emptyset, \zeta) \tilde{\Gamma}(\zeta; z) d\zeta d\eta.$$

From Lemma B.2, we know that there exists ε small enough such that, for any $\delta > 0$, there exists $B := B(\varepsilon, \delta) < \infty$ and $t_0 := t_0(\varepsilon, \delta) > 0$ such that, for all $0 < t < t_0$,

$$\sup_{z \in \mathbb{R}} \sup_{\eta \in [x+y-\delta, x+y+\delta]} \int p_t(\eta; \varepsilon, \emptyset, \zeta) \tilde{\Gamma}(\zeta; z) d\zeta \leq B.$$

The previous uniform bound for any η in a neighborhood of $x + y$ and all $z \in \mathbb{R}$ allows us to interchange the differentiation and the other relevant operations (integration, expectation, etc.) so that

$$\begin{aligned}\partial_y \mathbb{P}(X_t(x) \geq x + y | N_t^\varepsilon = n) &= \frac{n!}{t^n} \int_{\Delta} \partial_y \mathbb{P}(X_t(\varepsilon, \{s_1, \dots, s_n\}, x) \geq x + y) ds_n \dots ds_1 \\ &= \frac{n!}{t^n} \int_{\Delta} \mathbb{E} [\partial_y G_{t-s_n}(X_{s_n}(\varepsilon, \{s_1, \dots, s_{n-1}\}, x); x, y)] ds_n \dots ds_1 \\ &= \frac{n!}{t^n} \int_{\Delta} \mathbb{E} \left[\int p_{t-s_n}(x + y; \varepsilon, \emptyset, \zeta) \tilde{\Gamma}(\zeta; X_{s_n}(\varepsilon, \{s_1, \dots, s_{n-1}\}, x)) d\zeta \right] ds_n \dots ds_1\end{aligned}$$

and also, for any $0 < t < t_0$,

$$|\partial_y \mathbb{P}(X_t(x) \geq x + y | N_t^\varepsilon = n)| \leq B.$$

Using this bound, we finally have that

$$|\partial_y \bar{\mathcal{R}}_t(x, y)| \leq e^{-\lambda_\varepsilon t} \sum_{n=3}^{\infty} |\partial_y \mathbb{P}(X_t(x) \geq x + y | N_t^\varepsilon = n)| \frac{(\lambda_\varepsilon t)^n}{n!} \leq B e^{-\lambda_\varepsilon t} \sum_{n=3}^{\infty} \frac{(\lambda_\varepsilon t)^n}{n!} \leq B \lambda_\varepsilon^3 t^3.$$

The proof is then complete. ■

Proof of Lemma 6.1. By conditioning on the times of the jumps, which are necessarily distributed as the order statistics of n independent uniform $[0, t]$ random variables, we have

$$\mathbb{P}(|X_t - x| \geq \log y | N_t^\varepsilon = n) = \frac{n!}{t^n} \int_{\Delta} \mathbb{P}(|X_t(\varepsilon, \{s_1, \dots, s_n\}, x) - x| \geq \log y) ds_n \dots ds_1,$$

where $\Delta := \{(s_1, \dots, s_n) : 0 < s_1 < s_2 < \dots < s_n < t\}$. Hence, we only need to bound

$$\sup_{n \in \mathbb{N}, t \in [0, 1]} \frac{1}{n!} \int_0^\infty \mathbb{P}(|X_t(\varepsilon, \{s_1, \dots, s_n\}, x) - x| \geq \log y) dy$$

uniformly. By conditioning again,

$$\begin{aligned}\mathbb{P}(|X_t(\varepsilon, \{s_1, \dots, s_n\}, x) - x| \geq \log y) &= \mathbb{E} \left[\mathbb{P}(|X_t(\varepsilon, \{s_1, \dots, s_n\}, x) - x| \geq \log y | \mathcal{F}_{s_n}^-) \right] \\ &\leq \mathbb{E} \left[\mathbb{P}(|X_{t-s_n}(\varepsilon, \emptyset, z) - x| + |\gamma(z, J)| \geq \log y) \Big|_{z=X_{s_n}(\varepsilon, \{s_1, \dots, s_{n-1}\}, x)} \right].\end{aligned}$$

Recall the Condition **(C5)**, we have for some constant $M > 0$ and all $\lambda \leq 3$

$$\sup_x \mathbb{E} e^{\lambda |\gamma(x, J)|} \sup_x \leq C \int e^{|3\gamma(x, z)|} h(z) dz \leq M < \infty.$$

Now fix any positive constant A and $t \leq 1$, we have

$$\begin{aligned}\mathbb{E} e^{|X_t(\varepsilon, \{s_1, \dots, s_n\}, x) - x|} &= \int_0^\infty \mathbb{P}\{|X_t(\varepsilon, \{s_1, \dots, s_n\}, x) - x| > \log y\} dy \\ &= \int_0^A \mathbb{P}\{|X_t(\varepsilon, \{s_1, \dots, s_n\}, x) - x| > \log y\} dy \\ &\quad + \int_A^\infty \mathbb{P}\{|X_t(\varepsilon, \{s_1, \dots, s_n\}, x) - x| > \log y\} dy \\ &\leq A + \int_0^\infty \mathbb{E} \left[\mathbb{P}\{|X_{t-s_n}(\varepsilon, \emptyset, z) - x| + |\gamma(z, J)| \geq \log y\} \Big|_{z=X_{s_n}(\varepsilon, \{s_1, \dots, s_{n-1}\}, x)} \right] dy \\ &\leq A + 2e^{\frac{1}{2}\lambda_1^2 k(1+\exp(\lambda_1 \varepsilon))} \frac{1}{A^\alpha} \frac{1}{\alpha} \left(\mathbb{E} e^{\lambda_1 |X_{s_n}(\varepsilon, \{s_1, \dots, s_{n-1}\}, x) - x|} \right) \left(\mathbb{E} e^{\lambda_1 |\gamma(x, J)|} \right) \\ &\leq A + 2M e^{\frac{1}{2}\lambda_1^2 k(1+\exp(\lambda_1 \varepsilon))} \frac{1}{A^\alpha} \frac{1}{\alpha} \left(\mathbb{E} e^{\lambda_1 |X_{s_n}(\varepsilon, \{s_1, \dots, s_{n-1}\}, x) - x|} \right).\end{aligned}$$

Above, we used (17) for the last inequality with $\lambda = \lambda_1 = 1 + \alpha$, where $0 < \alpha < 2$ is to be chosen later. Now we iterate the above procedure by taking $\lambda_i = (1 + \alpha)^i, i = 1, 2, \dots, n$, at each step, and choose $\lambda_n = (1 + \alpha)^n = e$. We conclude that there exists a large enough constant C independent of n and t such that

$$\int_0^\infty \mathbb{P}\{|X_t(\varepsilon, \{s_1, \dots, s_n\}, x) - x| > \log y\} dy \leq C^n \left(\frac{1}{\alpha}\right)^n.$$

In what follows, we only need to show $C^n (1/\alpha)^n / n! \rightarrow 0$ as $n \rightarrow \infty$. Recall that $\alpha = e^{1/n} - 1$. We have

$$\log \left[C^n \left(\frac{1}{\alpha}\right)^n \right] \sim n \left(C + \log \frac{1}{n} \right) \quad \text{as } n \rightarrow \infty.$$

On the other hand, we know $\log n! \sim n^2/2$ as $n \rightarrow \infty$. The proof is then complete. ■

References

- [1] Y. Ait-Sahalia and J. Jacod. Testing for jumps in a discretely observed process. *Annals of Statistics*, 37:184-222, 2009.
- [2] Y. Ait-Sahalia and J. Jacod. Is Brownian motion necessary to model high frequency data? *Annals of Statistics*, 38:3093-3128, 2010.
- [3] D. Applebaum, *Lévy Processes and Stochastic Calculus*, Cambridge University Press, 2004.
- [4] O.E. Barndorff-Nielsen and N. Shephard. Econometrics of testing for jumps in financial economics using bipower variation. *Journal of Financial Econometrics*, 4(1):1-30, 2006.
- [5] H. Berestycki, J. Busca and I. Florent, Asymptotics and calibration of local volatility models, *Quantitative Finance*, 2:61-69, 2002.
- [6] H. Berestycki, J. Busca and I. Florent, Computing the Implied Volatility in Stochastic Volatility models, *Communications on Pure and Applied Mathematics*, Vol. LVII:1352-1373, 2004.
- [7] K. Bichteler, J.B. Gravereaux and J. Jacod: *Malliavin Calculus for Processes with Jumps*, Stochastics Monographs Volume 2. Gordon and Breach Science Publishers (1987).
- [8] P. Carr and L. Wu, What type of process underlies options? A simple robust test, *Journal of Finance*, 58(6):2581-2610, 2003.
- [9] R. Cont and P. Tankov, *Financial Modelling with Jump Processes*, Chapman & Hall, 2004.
- [10] J. Feng, M. Forde, and J.P. Fouque, Short maturity asymptotics for a fast mean reverting Heston stochastic volatility model, *SIAM Journal on Financial Mathematics*, 1:126-141, 2010.
- [11] J.E. Figueroa-López and M. Forde, The small-maturity smile for exponential Lévy models, *Submitted*, Available at arXiv:1105.3180v1 [q-fin], 2011.
- [12] J. Figueroa-López, and C. Houdré, Small-time expansions for the transition distribution of Lévy processes, *Stochastic Processes and their Applications*, 119:3862-3889, 2009.
- [13] J.E. Figueroa-López, R. Gong and C. Houdré, Small-time expansion of the distributions, densities, and option prices of stochastic volatility models with Lévy jumps, *Stochastic Processes and their Applications*, 122:1808-1839, 2012.
- [14] Forde, M., and Jacquier, Small-time asymptotics for an uncorrelated local-stochastic volatility model, *Appl. Math. Finance*, 18, 517-535, 2011.

- [15] Gao, K., and R. Lee, Asymptotics of Implied Volatility in Extreme Regimes, *Preprint*, Available at <http://papers.ssrn.com/>, 2011.
- [16] J. Gatheral, E. Hsu, P. Laurence, C. Ouyang and T-H. Wang, Asymptotics of implied volatility in local volatility models, *To appear in Mathematical Finance*, 2009.
- [17] Y. Ishikawa, Asymptotic behavior of the transitions density for jump type processes in small time, *Tôhoku Math. J.* 46:443-456, 1994.
- [18] Y. Ishikawa, Density estimate in small time or jump processes with singular Lévy measures, *Tôhoku Math. J.* 53:183-202, 2001.
- [19] R. Léandre, Densité en temps petit d'un processus de saut, In: *Seminaire de Probabilités XXI, Lecture Notes in Math. (J. Azéma, P.A. Meyer, and M. Yor, eds.)*, Springer, Berlin, 1247:81-99, 1987.
- [20] J. P. Lepeltier and R. Marchal Problèmes de martingales associées à un opérateur integro différentiel, *Ann. Inst. Henri Poincaré Probab. Stat.* B.12:43-103, 1976.
- [21] S. Levendorskii, American and European options in multi-factor jump diffusion model, near expiry, *Finance Stoch.*, 12:541-560, 2008.
- [22] P. Marchal. Small time expansions for transition probabilities of some Lévy processes. *Electronic communications in probability*, 14:132–142, 2009.
- [23] A. Medvedev and O. Scaillet, Approximation and calibration of short-term implied volatility under jump-diffusion stochastic volatility, *The review of Financial Studies*, 20(2):427-459, 2007.
- [24] J. Muhle-Karbe and M. Nutz, Small-time asymptotics of option prices and first absolute moments, *Preprint*, 2010.
- [25] D. Nualart. *The Malliavin Calculus and Related Topics*. Springer-Verlag, New York, USA, 1995.
- [26] B. Oksendal and A. Sulem: *Applied Stochastic Control of Jump Diffusions*, Springer (2005).
- [27] J. Picard, On the existence of smooth densities for jump processes, *Probab. Theory Related Fields*, 105:481-511, 1996.
- [28] J. Picard, Density in small time at accessible points for jump processes, *Stochastic processes and their applications*, 67:251-279, 1996.
- [29] M. Podolskij. *New Theory on estimation of integrated volatility with applications*. PhD thesis, Ruhr-Universität Bochum, April 2006.
- [30] P. Protter. *Stochastic Integration and Differential Equations*. Springer, 2004. 2nd Edition.
- [31] M. Roper, Implied volatility: small time to expiry asymptotics in exponential Lévy models. *Thesis, University of New South Wales*, 2009.
- [32] L. Rüschendorf and J. Woerner. Expansion of transition distributions of Lévy processes in small time. *Bernoulli*, 8:81–96, 2002.
- [33] I. Shigekawa: *Stochastic Analysis*, Translations of Mathematical Monographs Volume 224. AMS (2004).
- [34] P. Tankov, Pricing and hedging in exponential Lévy models: review of recent results, *To appear in the Paris-Princeton Lecture Notes in Mathematical Finance, Springer 2010*.
- [35] J. Yu, Closed-form likelihood approximation and estimation of jump-diffusions with an application to the realignment risk of the Chinese Yuan. *Journal of Econometrics*, 141:1245-1280, 2007.